Notes on the discrete Reifenberg theorem of Naber-Valtorta

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1 basic definitions

We live in R^n, and fix a $k \leq n$. We let d_H be the usual Hausdorff distance between sets. For any two subpaces L_1, L_2, we defined the Grassmanian distance

$$d_G(L_1, L_2) := d_H(L_1 \cap B_1, L_2 \cap B_1).$$

If L_1, L_2 are affine spaces, we define d_G in terms of the associated subspaces.

Given a function f I define

$$|f|_{C^k_\rho} = \sum_{\ell=0}^{k} \sup_{\mathrm{dom}(f)} \rho^{\ell-1} |D^\ell f|.$$

and $|f|_{C^k_\rho \Omega} = |f|_\Omega|_{C^1_\rho}$.

Given an affine k-space L, writing $\mathrm{graph}_L(g)$ means $\{(x, g(x)) : x \in \Omega\}$ for some $g : \Omega \to L^\perp$ and some $\Omega \neq \emptyset$. We disallow vacuous graphs.

Given a measure μ, define the rescaled measure at $B_\rho(y)$ by

$$\mu_{y,\rho}(A) := \rho^{-k} \mu(y + \rho A).$$

If we say "apply theorem X to μ at scale $B_\rho(y)$", what we mean is "apply theorem X to the rescaled measure $\mu_{y,\rho}$".

2 Reifenberg's disc theorem

As a warm up we re-examine Reifenberg's original disc theorem. I would highly recommend understanding the proof, and in particular the construction of the approximating manifolds, before reading discrete Reifenberg.

This section is basically lifted off Leon's notes (REF), with some simplifications.

Theorem 2.1 (Reifenberg's disc theorem)**.** *Let $S \subset R^n$ be a closed set, satisfying the following property: $\exists \delta$, so that for any $x \in S \cap B_1$, and any $r \in (0, 8)$, we have a k-dimensional affine plane $L_{x,r} \ni x$ so that*

$$d_H(S \cap B_r(x), L_{x,r} \cap B_r(x)) \leq \delta r.$$

Then, provided $\delta \leq \delta_0(k, n, \alpha)$, $S \cap B_1$ is bi-Holder equivalent to a k-disc, with Holder exponent α.

2.2 squash lemma

The key lemma, which we will use throughout this exposition, is

Lemma 2.3 (squash lemma)**.** *Suppose G is some set in R^n, and L an affine k-plane, so that*

$$\begin{cases} G \cap B_2 = \mathrm{graph}_L f \\ |f|_{C^1_1} \leq 1 \end{cases}.$$

Let $\Phi(x) = m + p_L(x-m) + e(x)$*, where:* p_L *is the linear projection to the subspace associated to* L*;* $m \in B_2$ *is a point satisfying* $\mathrm{dist}(m, L) \leq \epsilon$*; and* $e : B_2 \to B_2$ *is some* C^1 *function satisfying* $|e|_{C^1_1} \leq \epsilon$*.*

Then for ϵ *sufficiently small, depending only on* n*, we have*

$$\begin{cases} \Phi(G \cap B_2) = \mathrm{graph}_L \tilde{f} \\ |\tilde{f}|_{C^1_1} \leq c(n)\epsilon. \end{cases}$$

Remark 2.4. Notice that the assumptions and conclusion of the squash lemma are scale-invariant.

Proof. By shifting f, $\tilde{f}$ by ϵ, we can reduce to the case $m \in L$. Also, by moving the origin WLOG $L = R^k \times \{0\}$. Write

$$(e_1(x), e_2(x)) = (p_L(e(x, f(x))), p_{L^\perp}(e(x, f(x)))),$$

and let $U = p_L(G \cap B_2)$. Define

$$g : U \to L : x \mapsto x + e_1(x).$$

Notice that $g(U) = p_L(\Phi(G \cap B_2))$.

By assumption we have $|g - Id|_{C^1_1(U)} \leq c(n)\epsilon$, and so (provided $\epsilon \leq \epsilon_0(n)$) there is a C^1 inverse $g^{-1} : g(U) \to U$, with $|g^{-1} - Id|_{C^1_1} \leq c(n)\epsilon$.

Now given $x \in U$, write

$$\begin{aligned} \Phi(x, f(x)) &= (x + e_1(x), e_2(x)) \\ &= (g(x), e_2(g^{-1}(g(x))). \end{aligned}$$

Therefore $\Phi(G \cap B_2) = \mathrm{graph}_L(e_2 \circ g^{-1})$, with $|e_2 \circ g^{-1}|_{C^1_1} \leq c(n)\epsilon$. □

2.5 approximating manifolds

Here's the strategy of the proof: we cook up a sequence of approximating manifolds M_i, which converge in the Hausdorff sense to S, and which have good graphical control at scale r_i.

We shall fix our scale $r_i = 8^{-i}$. For each $i \geq 0$, we let $\{y_{ij}\}_{j=1}^{Q_i}$ be a maximal $r_i/2$-separated set in $S \cap B_1$, and define $L_{ij} = L_{y_{ij}, 8r_i}$. Let $p_{ij} = \mathrm{proj}_{L_{ij}}$ and $p_{ij}^\perp = \mathrm{proj}_{L_{ij}^\perp}$ be the associated linear projections.

Take ϕ_{ij} a POU such that:

$$\begin{aligned} &\mathrm{spt}\phi_{ij} \subset B_{3r_i}(y_{ij}), \\ &\sum_j \phi_{ij} = 1 \text{ on } \cup_j B_{5r_i/2}(y_{ij}) \\ &|\nabla \phi_{ij}| \leq 10/r_i. \end{aligned}$$

Define

$$\sigma_i(x) = x - \sum_j \phi_{ij}(x) p_{ij}^\perp(x - y_{ij}).$$

Set $M_0 = L_{0,8}$, and $M_i = \sigma_i(M_{i-1})$. These are our approximating manifolds.

We seek to prove the following inductive hypothesis:

$$(\star_i) \begin{cases} M_i \cap B_{1+r_i} \subset \cup_j B_{2r_i}(y_{ij}) \\ M_i \cap B_{2r_i}(y_{ij}) = \mathrm{graph}_{L_{ij}} g \\ |g|_{C^1_{r_i}} \leq \Lambda(n)\delta \end{cases}$$

Here Λ is some constant to be determined.

We prove two estimates: "coarse estimates" and "squash estimates". Combining these will yield $(\star_i)$.

Lemma 2.6 ("coarse: give up bounds, improve radius")**.** *Suppose*

$$\begin{cases} M_{i-1} \cap B_{2r_{i-1}}(y_{i-1,j}) = \mathrm{graph}_{L_{i-1,j}}(g) \\ |g|_{C^1_{r_{i-1}}} \leq \beta\delta. \end{cases}$$

Then

$$\sup_{M_{i-1}} \left(r_i^{-1}|\sigma_i - Id| + |D^\perp_{M_{i-1}}(\sigma_i - Id)| + |D^\top_{M_{i-1}}(\sigma_i - Id)|^{1/2} \right) \leq c_3(n)(1+\beta)\delta.$$

In particular, provided $\delta \leq \delta_2(n, \beta)$, we have

$$\begin{cases} M_i \cap B_{3r_i}(y_{ij}) = \mathrm{graph}_{L_{ij}}(g) \\ |g|_{C^1_{r_i}} \leq c_4(n,\beta)\delta. \end{cases},$$

Remark 2.7. Notice we have a power improvement for tangential derivatives! Although this is irrelevant for the classical Reifenberg theorem, this is very important in discrete Reifenberg: essentially it means that if we move Hausdorff distance d, then our volume changes by only d^2.

Proof. We have

$$\sigma_i(x) - x = -\sum_j \phi_{ij}(x) p^\perp_{ij}(x - y_{ij}).$$

Recall that $\sigma_i = id$ outside $\cup_j B_{3r_i}$. Fix an $j \in \{1, \ldots, Q_i\}$. Then $y_{ij} \in B_{r_{i-1}}(y_{i-1,j_0})$ for some $j_0 \in \{1, \ldots, Q_{i-1}\}$, and hence

$$B_{3r_i}(y_{ij}) \subset B_{2r_i}(y_{i-1,j_0}).$$

So $B_{3r_i}(y_{ij}) \cap M_{i-1} = \mathrm{graph}_{L_{i-1,j_0}}(g)$ with good bounds on g. Take $x \in B_{3r_i}(y_{ij}) \cap M_{i-1}$. If $\phi_{ik}(x) > 0$, then necessarily $|y_{ik} - y_{ij}| \leq 6r_i$, and we have

$$\begin{aligned} &\left|p^\perp_{ik}(x - y_{ik})\right| \\ &= \left|p^\perp_{i-1,j_0}(x - y_{i-1,j_0}) + p^\perp_{i-1,j_0}(y_{i-1,j_0} - y_{ik}) + (p^\perp_{ik} - p^\perp_{i-1,j_0})(x - y_{ik})\right| \\ &\leq |g(p_{i-1,j_0}(x))| + \mathrm{dist}(y_{ik}, L_{i-1,j_0}) + d(L_{ik}, L_{i-1,j_0})|x - y_{ik}| \\ &\leq \beta\delta r_{i-1} + c(n)\delta r_{i-1} + c(n)\delta r_i \\ &\leq c(n)(1+\beta)\delta r_i. \end{aligned}$$

since $y_{ik} \in S$, we can use lemma reifenberg-conseq to bound the second and third terms.

But now $|y_{ij} - y_{ik}| \geq r_i/2$, and so there are at most $\tilde{c}(n)$ integers k with $\phi_{ik}(x) > 0$. Thus

$$|\sigma_i(x) - x| \leq \tilde{c}c(1+\beta)\delta r_i.$$

We prove the gradient bound. For any vector V, we have

$$D_V(\sigma_i - Id) = -\sum_j (D_V \phi_{ij}) p_{ij}^\perp (x - y_{ij}) - \sum_j \phi_{ij} p_{ij}^\perp (V).$$

Take y_{ij}, y_{i-1,j_0}, and $x \in B_{3r_i}(y_{ij})$ as above. Choose any y_{ik} so that $\phi_{ik}(x) > 0$. By the previous argument, with our definition of the ϕ_{ij}, we have

$$|D_V \phi_{ik}||p_{ik}^\perp (x - y_{ik}| \leq c(n)(1+\beta)\delta.$$

The second term can be bounded as:

$$\begin{aligned} |p_{ik}^\perp (V)| &\leq |(p_{ik}^\perp - p_{i-1,j_0}^\perp)(V)| + |p_{ik}^\perp(V)| \\ &\leq cd(L_{ik}, L_{i-1,j_0}) + cd(L_{i-1,j_0}, T_x M_{i-1}) \\ &\leq c(1+\beta)\delta. \end{aligned}$$

The bound on $|D_{M_{i-1}}^\perp(\sigma_i - Id)|$ now follows precisely as before.

We consider $D^\top$. Take a unit vector $W \in T_x M_{i-1}$. Then for any vector Y, we have

$$\begin{aligned} |W \cdot p_{ij}^\perp(Y)| \leq |p_{ij}^\perp(W)||p_{ij}^\perp(Y)| &\leq \left(|p_{ij}^\perp - p_{i-1,j_0}^\perp| + |p_{i-1,j_0}^\perp(W)|\right) |p_{ij}^\perp(Y)| \\ &\leq c(1+\beta)\delta |p_{ij}^\perp(Y)|. \end{aligned}$$

Therefore we have the improved bound

$$|W \cdot D_V(\sigma_i - Id)| \leq c(1+\beta)^2\delta^2. \qquad \square$$

Lemma 2.8 ("squash: give up radius, improve bounds")**.** *Suppose*

$$\left\{ \begin{array}{l} M_{i-1} \cap B_{2r_{i-1}}(y_{i-1,j}) = \mathrm{graph}_{L_{i-1,j}}(g) \\ |g|_{C^1_{r_{i-1}}} \leq \alpha\delta \end{array} \right. .$$

Then provided $\delta \leq \delta_3(n, \alpha)$, we have

$$\left\{ \begin{array}{l} M_i \cap B_{2r_i}(y_{ij}) = \mathrm{graph}_{L_{ij}}(g) \\ |g|_{C^1_{r_i}} \leq c_2(n)\delta \end{array} \right. .$$

Proof. Fix a $y_1 = y_{i,j_1}$, and write $L_1 = L_{i,j_1}$, $p_1 = p_{i,j_1}$. We have

$$\begin{aligned} \sigma_i(x) &= y_1 + p_1(x - y_1) + p_1^\perp(x - y_1) - \sum_j \phi_{ij}(x) p_{ij}^\perp (x - y_{ij}) \\ &=: y_1 + p_1(x - y_1) + e(x). \end{aligned}$$

For $x \in B_{5r_i/2}(y_1)$, since $\sum_j \phi_{ij} = 1$, we can write

$$\begin{aligned} e(x) &= \sum_j \phi_{ij}(x) \left(p_1^\perp(x - y_1) - p_{ij}^\perp(x - y_{ij}) \right) \\ &= \sum_j \phi_{ij}(x) \left(p_1^\perp(y_{ij} - y_1) + (p_0^\perp - p_{ij}^\perp)(x - y_{ij}) \right). \end{aligned}$$

We deduce that $|e|_{C^1_{r_i} B_{5r_i/2}(y_0)} \leq c_5(n)\delta$.

We'd like to apply lemma squash. We can find an j_0 so that $y_1 \in B_{r_{i-1}}(y_{i-1,j_0})$. We have $M_{i-1} \cap B_{2r_{i-1}}(y_{i-1,j_0}) = \mathrm{graph}_{L_{i-1,j_0}}(g)$, with $|g|_{C^1_{r_{i-1}}} \leq \alpha\delta$.

We know $B_{2r_j} \subset B_{2r_{j-1}}(y_{j-1,i_0})$ and $d(L_1, L_{j-1,i_0}) \leq 32\delta$. So provided $\alpha\delta \leq 1/32$, and 32δ is sufficiently small (depending only on n), then $M_{i-1} \cap B_{2r_i}(y_1) = \mathrm{graph}_{L_1}(\tilde{g})$, with $|\tilde{g}|_{C^1_{r_i}} \leq 1$.

Apply lemma squash at scale $B_1 \equiv B_{5r_i/2}(y_1)$ to deduce

$$\begin{cases} \sigma_i(M_{i-1} \cap B_{3/2r_i}) = \mathrm{graph}_{L_1} g \\ |g|_{C^1_{r_i}} \leq \tilde{c}_5\delta \end{cases}$$

By the coarse estimates we also know $|\sigma_i(x) - x| \leq c_3(1+\alpha)\delta r_i$. Thus if $c_3(1+\alpha)\delta \leq 1/32$ we can replace $\sigma_i(M_{i-1} \cap B_{5r_i/2})$ with $\sigma_i(M_{i-1}) \cap B_{2r_i}$. □

By combining the two we have entire control over σ_i.

Lemma 2.9 ("improved coarse estimates"). *Suppose* $(\star)$ *holds through* $i-1$, *then*

$$\sup_{M_{i-1}} \left(r_i^{-1}|\sigma_i - id| + |D_{M_{i-1}}(\sigma_i - id)| \right) \leq c_3(1+c_2)\delta.$$

Proof. If $i > 1$ apply the squash estimates at scale $i-2$ to deduce graphical control of M_{i-1} with bound $\leq c_2\delta$. If $i = 1$ we have this bound on $M_{i-1} = M_0$ trivially. Now plug this into the coarse estimates at scale $i-1$. □

Theorem 2.10. *We have* $(\star_i)$ *for all* i.

Proof. Trivially $(\star_0)$ holds. Suppose $(\star_{i-1})$ holds. Since $M_{i-1} \cap B_{1+r_{i-1}}$ lies entirely within the region in which we have good graphical control, we know that in fact

$$M_{i-1} \cap B_{1+r_{i-1}} \subset B_{c(n)\delta r_{i-1}}(S).$$

Hence, choosing $\Lambda\delta$ sufficiently small (depending only on n), we have

$$M_{i-1} \cap B_{1+r_i} \subset \cup_j B_{3r_i/2}(y_{ij}).$$

The coarse estimates then imply $M_i \cap B_{1+r_i} \subset \cup_j B_{2r_i}(y_{ij})$.

For graphicality we require $\Lambda \geq c_2$. Now apply the squash estimates, ensuring that δ is sufficiently small, depending only on n, k, Λ. □

Theorem 2.11. *We have that*

$$d_H(M_i \cap B_1, S \cap B_1) \leq c_6(n)\delta r_i.$$

Proof. The fact that $M_i \cap B_{1+r_i} \subset B_{c_6\delta r_i}(S)$ follows trivially from the inductive hypothesis.

To prove $S \subset B_{c_6\delta r_i}(M_i)$, observe that $(\star_i)$ implies

$$\mathrm{dom}(g) \supset B_{3r_i/2}(y_{ij}) \cap L_{ij}.$$

This follows because $\mathrm{dom}(g) \neq \emptyset$, and of course provided $\Lambda\delta$ is sufficiently small (depending on n). Since $\{B_{r_i}\}$ cover $S \cap B_1$, this evidently implies the required bound. □

2.12 the map itself

Here we construct the actual bi-Holder map $\overline{B}_1 \to S$.

Lemma 2.13. *For $x, y \in M_{i-1}$, we have*

$$|(\sigma_i(x) - \sigma_i(y)) - (x - y)| \leq c_7(n)\delta|x - y|.$$

Proof. If $|x - y| \geq r_i$ this follows directly from the coarse estimates:

$$\begin{aligned} |(\sigma_i(x) - \sigma_i(y)) - (x - y)| &\leq |\sigma_i(x) - x| + |\sigma_j(y) - y| \\ &\leq c(n)\delta r_i \\ &\leq c(n)|x - y|. \end{aligned}$$

So we can suppose $|x - y| < r_i$. Write

$$\sigma_i(x) - \sigma_i(y) - (x - y) = \sum_j \phi_{ij}(x)p_{ij}^\perp(x - y) + (\phi_{ij}(x) - \phi_{ij}(y))p_{ij}^\perp(y - y_{ij}).$$

If $\phi_{ij}(x) > 0$, then $x, y \in B_{4r_j}(y_{ji}) \subset B_{2r_{i-1}}(y_{i-1,j_0})$ for some j_0. Then $(\star_i)$ implies

$$\begin{aligned} |p_{ij}^\perp(x - y)| &\leq |(p_{ij}^\perp - p_{i-1,j_0}^\perp)(x - y)| + |p_{i-1,j_0}^\perp(x - y)| \\ &\leq c(n)\delta|x - y| + |g(p_{ij}(x)) - g(p_{ij}(y))| \\ &\leq c(n)\delta|x - y|. \end{aligned}$$

Of course for any give x, there are at most $\tilde{c}(n)$ integers i so that $\phi_{ij}(x) > 0$. Thus we've bounded the first term.

Similarly, if either $\phi_{ij}(x) > 0$ or $\phi_{ij}(y) > 0$, we know $|y - y_{ij}| < 4r_i$, and so the second bound follows identically. □

We define $\tau_i(x) = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_1$. By the coarse estimates,

$$\begin{aligned} |\tau_{i+k}(x) - \tau_i(x)| &\leq \sum_{\ell=1}^{k} |\sigma_{i+\ell}(\tau_{i+\ell-1}(x)) - \tau_{i+\ell-1}(x)| \\ &\leq c_3(1 + \Lambda)\delta \sum_{\ell=1}^{\infty} r_{i+\ell+1} \\ &\leq \frac{1}{2} 8^{-i}, \end{aligned}$$

having chosen δ small. So in $\overline{B_1}$, $\{\tau_i\}$ are uniformly C^0-Cauchy, with some C^0 limit $\tau : \overline{B_1} \to R^n$.

Combing this with the above lemma, we obtain for *any* i that

$$\begin{aligned}|\tau(x) - \tau(y)| &\leq |\tau(x) - \tau_i(x)| + |\tau_i(x) - \tau_i(y)| + |\tau_i(y) - \tau(y)| \\ &\leq 8^{-i} + (1 + c_7\delta)^i |x - y|.\end{aligned}$$

Fix an $\alpha \in (0, 1)$. Now for any $|x - y| < 1/2$ we can pick an i so that

$$8^{-i} \leq |x - y|^\alpha, \quad (1 + c_7\delta)^i \leq |x - y|^{\alpha - 1}.$$

In detail, we need

$$i \geq \frac{\alpha}{\log 8} \log\left(|x - y|^{-1}\right), \quad i \leq \frac{1 - \alpha}{\log(1 + c_7\delta)} \log\left(|x - y|^{-1}\right),$$

which can certainly be achived provided δ is sufficiently small, depending on n, α. Therefore we obtain that

$$|\tau(x) - \tau(y)| \leq 2|x - y|^\alpha.$$

Similarly, we use the above lemma to obtain, for any i, that

$$|x - y| \leq (1 + c_7\delta)^i (8^{-i} + |\tau(x) - \tau(y)|).$$

Ensuring δ is sufficiently small, we can (for each $|x - y| < 1/2$) chose a i so that

$$\frac{1}{2}|x - y|^\alpha \leq |\tau(x) - \tau(y)|.$$

This proves the theorem.

3 discrete Reifenberg

We use Greek indices to indicate the scale $r_\alpha = 2^{-\alpha}$.

For an arbitrary Radon measure μ define, for $\epsilon_k \leq \omega_k 320^{-k} \cdot 4^{-k}$,

$$D^k_\mu(x, r) = \begin{cases} \inf_{L^k} r^{-k-2} \int_{B_r(x)} \mathrm{dist}(z, L)^2 d\mu(z) & \mu B_r(x) \geq \epsilon_k r^k \\ 0 & otherwise. \end{cases}$$

where the infimum is taken over *affine* k-planes L. We set $V^k_\mu(x, \rho)$ to be k-plane realizing this infimum. We simply write D and V where there is no confusion. Notice that D is scale-invariant, in the sense that

$$D^k_{\mu_{y,\rho}}(x, r) = D^k_\mu(y + \rho x, \rho r).$$

We work towards proving the following theorem.

Theorem 3.1 (discrete reifenberg)**.** *Let $\{B_{r_p}(x_p)\}_p$ be a collection of disjoint balls with $r_p \leq 1$. Set*

$$\mu = \sum_p r_p^k \delta_{x_p}.$$

Suppose for any $x \in B_2$, and any $\alpha_0 \in \{0, 1, 2, \ldots\}$, we knoew

$$\sum_{\alpha \geq \alpha_0} \int_{B_{2r_{\alpha_0}}(x)} D_\mu^k(z, 16r_\alpha) d\mu(z) \leq r_{\alpha_0}^k \delta^2.$$

Then, provided $\delta \leq \delta_0(n)$, we have

$$\mu B_1 = \sum_{x_p \in B_1} r_p^k \leq C_{dr}(n).$$

3.2 intuition/motivation

Notice the discrete-Reifenberg theorem is trivial if $k = n$, simply because μ measures the total volume of a bunch of disjoint balls, and so $\mu B_1 \leq \omega_n 2^n$. The issue arises when $k < n$, essentially because at any fixed scale the number of disjoint balls grows like ρ^{-n}, but measure only grows like ρ^k.

However, if D is small, then everything a fixed distance away from the L^2-best plane $V(0,1)$ has negligable μ measure. In fact the assumption on D implies that the *total* excess, over all scales, is negligable. Ensuring $D(0,1) \leq \delta(\rho)$, all the original ρ-balls must be clustered near $V(0,1)$. This says the number of ρ-balls only grows like ρ^{-k}.

So what if we try this strategy: extend the original ρ-balls to a Vitali covering of $V(0,1)$, which we divide into "good" balls of measure $\geq \rho^k$, "bad" balls of measure $< \rho^k$, and the "original" ρ-balls. Away from these balls we have neglibable measure.

Since all these balls are clustered near $V(0,1)$, if we cut out holes of size $balls/5$ from $V(0,1)$, then these holes are disjoint. So basically

$$\#\{good, bad, original\} \leq (2^k)/(\rho/5)^k = 10^k \rho^{-k}.$$

Now let's do this again in each good ball. Suppose, for argument's sake, we don't have original balls of scale $< \rho^2$. Then our process terminates in the good ρ-balls, and in particular we have

$$\mu B_\rho(good) \leq \rho^{2k} \cdot (2\rho)^k/(\rho^2/5)^k + excess = 10^k \rho^k + exc.$$

Putting this together, we obtain

$$\begin{aligned}
\mu B_1 &\leq \sum_{bad, original, good} \mu B_\rho + exc \\
&\leq \#\{bad\}\rho^k + \#\{original\}\rho^k + \#\{good\}10^k\rho^k + exc \\
&\leq \#\{good, bad, original\}10^k \rho^k + exc \\
&\leq 10^{2k} + exc.
\end{aligned}$$

If we had to repeat this process for A steps, we would have a bound like

$$\mu B_1 \leq 10^{Ak} + exc.$$

This isn't very good, but is better than the stupidest possible bound $\sim \rho^{A(k-n)}$.

What is happening is that the exponential error 10^k encapsulates our "double-counting" on overlaps in the cover. We look at each good ball in isolation, and forget that actually most of the stuff inside is covered by other good/bad/original balls.

To eliminate double-counting we need a global memory of these balls. One is tempted to just use the original $V(0,1)$, and cut holes at all scales out of this plane. However you have the fact that $V(0,1)$ and $V(x,\rho)$ don't coincide, and somehow would need to map the potential holes in $V(x,\rho) \sim (otherballs)$ back to $V(0,1)$. On good balls the L^2-affine condition gives tilting control on the L^2-planes, but like in Reifenberg's theorem the net tilt may not be finite as your scale goes to 0.

So what we do is "interpolate" between the L^2-best planes, a la Reifenberg, to obtain a manifold associated to each scale, from which we can cut out holes and keep track of volume. It turns out to be somewhat delicate, partly because in order to get tilting control we need to assume volume control at lower scales. So we ultimately need to do some kind of double-induction, where we pull upper volume control up scale-by-scale, using this downwards hole-cutting technique at each scale.

Here is are some examples, courtesy of Aaron Naber, which motivate why these manifolds should have uniformly controlled volume. First consider two curves: γ_1 and γ_2. γ_1 is the line segment $[-1,1] \times \{0\}$. γ_2 is the piece-wise continuous curve obtain by connecting the points

$$(-1,0) \to (-1/2,0) \to (0,\epsilon) \to (1/2,0) \to (1,0).$$

Now we have $d_H(\gamma_1,\gamma_2) \leq \epsilon$, but the lengths differ by an improved amount $|L\gamma_1 - L\gamma_2| \leq 2\epsilon^2$.

Similarly, let γ_i be a sequence of Koch snowflakes, so that $d_H(\Gamma_i,\Gamma_{i-1}) \leq \epsilon_i \to 0$. These are defined in the usual way, where γ_0 is a line segment, then given γ_i we transform each straight edge in the fashion pictured above. By the above reasoning we have $|L\gamma_i - L\gamma_{i-1}| \leq c\epsilon_i^2$.

So to control $L\gamma_i$, it's enough to control the total *squared* distance $\sum \epsilon_i^2$, rather than the naive $\sum \epsilon_i$. This is why we can get away with a bound on $\sum_\alpha \int D$. Precisely, the phenomena described above manifests itself in the improved tangential derivative bounds on the Reifenberg maps σ_i, allowing us to squeeze out an extra power in the Jacobian.

4 general tilting control

We prove that good control on D, plus good control on volume, gives good tilting control.

Observation 4.1. *If $\mu B_r(x) \geq 2^k \epsilon_k r^k$, then*

$$D(x,r) \leq 2^{k+2} \fint_{B_r(x)} D(z,2r) d\mu \leq \frac{c_k}{r^k} \int_{B_r(x)} D(z,2r) d\mu(z).$$

This follows because for any $z \in B_r(x)$, we have

$$D(x,r) \leq 2^{k+2} \inf_{L^k} (2r)^{-k-2} \int_{B_{2r}(z)} \mathrm{dist}(y,L)^2 d\mu(y).$$

Therefore, the assumptions of theorem discrete-reifenberg imply that

$$D(x, 8r_\alpha) \leq c_k r_\alpha^2 \delta^2.$$

Lemma 4.2. *Take $B_r(y) \subset B_R(x)$ both with density ratio $\geq \epsilon_k$. Let $m_r(y)$ be the center of mass of $B_r(y)$. Then*

$$\begin{aligned} d(m_r(y), V(x,R))^2 &\leq \frac{1}{\mu B_r(y)} \int_{B_r(y)} d(z, V(x,R))^2 d\mu(z) \\ &\leq \frac{1}{\epsilon_k r^k} R^{k+2} D(x,R) \\ &\leq c(k)(R/r)^k R^2 D(x,R). \end{aligned}$$

Similarly, if we only assume instead that $\mu B_r(y) \geq m$, then we have the bound

$$d(m_r(y), V(x,R)) \leq \frac{R^k}{m} R^2 D(x,R).$$

Lemma 4.3. *There is a $\rho_0(M)$ so that if $\rho \leq \rho_0$, and $\mu B_1 \geq \epsilon_k$, and*

$$x \in B_1 \implies \mu B_\rho(x) \leq M\rho^k,$$

then for any $\leq k-1$ affine plane $W \subset B_1$, we can find an $x \in B_1$ so that

$$\mathrm{dist}(x,W) \geq 10\rho, \quad \mu B_\rho(x) \geq m_{avm}(M,\rho) > 0.$$

Proof. WLOG we can of course assume $\dim W = k-1$. Cover $B_{10\rho}(W) \cap B_1$ with balls $B_\rho(x_i)$ centered in B_1, so that $\{B_{\rho/2}(x_i)\}$ are disjoint. Then

$$\#\{x_i\} \leq c(n,k) \frac{\rho^{n-k+1}}{\rho^n} \leq c(n,k)\rho^{1-k}.$$

And so

$$\mu B_{10\rho}(W) \leq c(n,k) M \rho.$$

Suppose, towards a contradiction, that $\mu B_\rho(x) < m$ for every $x \in B_1 \sim B_{11\rho}(W)$. Cover B_1 by balls $B_\rho(y_i)$, centered in B_1, so that $\{B_{\rho/2}(y_i)\}$ are disjoint. Then of course $\#\{y_i\} \leq c(n)\rho^{-n}$, and therefore

$$\mu\left(B_1 \sim B_{10\rho}(W)\right) \leq c(n) m \rho^{-n}.$$

But since $\mu B_1 \geq \epsilon_k$, we have a contradiction by choosing $m \sim \rho^n$, and ρ sufficiently small, depending on M. □

The general tilting lemma says:

Lemma 4.4. *Let $\mu B_1 \geq 8^k \epsilon_k$. Suppose for some $x \in B_1$ we have the properties that $\mu B_\rho(x) \geq 8^k \epsilon_k \rho^k$, and*

$$y \in B_\rho(x) \implies \mu B_{\rho^2}(y) \leq M \rho^{2k}.$$

Suppose further that $d(x, A) \leq \rho/2$.

Then provided $\rho \leq \rho_0(M)$, we have

$$d_H \left(V(0,8) \cap B_\rho(x), V(x, 8\rho) \cap B_\rho(x)\right)^2 \leq c_{\text{tilt}}(n, M, \rho)(D(0,8) + D(x, 8\rho)).$$

Remark 4.5. Although we won't use it, as a side note we could alternately assume $d(x, B) \leq \rho/2$. (replacing $\rho/2$ with $\beta\rho$ for any $\beta < 1$ would also be fine). Or generally, assuming lower volume control like

$$\mu B_{\rho/2}(x) \geq c(k)\epsilon_k \rho^k,$$

would imply either of these hypothesis.

Proof. By ensuring c_{tl} is sufficiently big, we can of course assume

$$D(0,8) + D(x, 8\rho) < \delta(n, M, \rho).$$

For notational simplicity write $A = V(0,8)$, and $B = V(x, 8\rho)$. Let P_A be the *affine* projection onto A.

We wish to find a set of $k+1$ points $\{y_0, \ldots, y_k\} \subset B_\rho(x)$ so that $\{P_A(y_i)\}$ are in "general position", and we have the lower volume control $\mu B_{\rho^2}(y_i) \geq m_{avm}(\rho, M)\rho^k$. Here's how we do it.

Setting $W = \emptyset$, obtain a y_0 from lemma volumes-avoid-space at scale $B_1 \equiv B_\rho$. Then given $\{y_0, \ldots, y_{i-1}\}$, apply lemma volumes-avoid-space with

$$\begin{aligned} W_{i-1} &= P_A(m_0) + \text{span}\{P_A(m_1) - P_A(m_0), \ldots, P_A(m_{i-1}) - P_A(m_0)\} \\ &= \text{the affine } (i-1)\text{-space containing } P_A(m_0), \ldots, P_A(m_{i-1}). \end{aligned}$$

to obtain y_i. Here m_i is the μ-center of mass of $B_{\rho^2}(y_i)$. The lower volume bound is clearly satisfied.

Before we prove a general position principle let us observe that by lemma COM, we have

$$\text{dist}(m_i, A)^2 \leq c(n, \rho, M) D(0,8), \quad \text{dist}(m_i, B)^2 \leq c(n, \rho, M) D(x, 8\rho).$$

In particular, we have

$$d(P_A(m_i), B)^2 \leq 2d(m_i, A)^2 + 2d(m_i, B)^2 \leq c(n, \rho, M)(D(0,8) + D(x, 8\rho)).$$

I claim that, for every i,

$$P_A(m_i) \notin B_{5\rho^2}(W_{i-1}). \tag{1}$$

For $i = 0$ there is nothing to prove. Suppose it holds up to $p_A(m_{i-1})$. From lemma volumes-avoid-space, we have

$$\mathrm{dist}(y_i, W_{i-1} \cap B_\rho(x)) \geq 10\rho^2.$$

Since $\mathrm{dist}(x, A) \leq \rho/2$ and $W_{i-1} \subset A$, this implies

$$\mathrm{dist}(y_i, W_{i-1}) \geq 9\rho^2,$$

and therefore trivially we have $\mathrm{dist}(m_i, W_{i-1}) \geq 8\rho^2$.

Putting these together, we therefore obtain

$$\begin{aligned} d(P_A(m_i), W_{i-1}) &\geq d(m_i, W_{i-1}) - d(m_i, A) \\ &\geq 8\rho^2 - c(n, \rho, M)\delta \\ &\geq 5\rho^2 \end{aligned}$$

provided δ is sufficiently small (depending on n, ρ, M). This proves the claim.

Now given any $y \in A \cap B_\rho(x)$, since $\{P_A(m_i)\}$ are in general position (in the sense of EQREF), we can write

$$y = P_A(m_0) + \sum_{i=1}^{k} \beta_i \left(P_A(m_i) - P_A(m_0)\right)$$

with $|\beta_i| \leq c(\rho, M)|y - P_A(m_0)| \leq c(\rho, M)$.

Using this, we deduce

$$\begin{aligned} d(y, B) &\leq d(P_A(m_0), B) + \sum_{i=1}^{k} |\beta_i| d(P_A(m_i) - P_A(m_0), B) \\ &\leq c(n, \rho, M)(D(0, 8) + D(x, 8\rho))^{1/2}. \end{aligned}$$

Consequently, $A \cap B_\rho(x)$ lies within a $c(\rho, M)(D(0, 8) + D(x, 8\rho))^{1/2}$ neighborhood of B. Since $d(x, A) \leq \rho/2$, this will imply (for δ suffiently small) the required Hausdorff distance bound. □

5 downwards discrete Reifenberg

We shall now fix our scale $r_i = \rho^i$, where $\rho = 2^{-\beta_0} \leq \min\{\rho_0(C_1), 1/10\}$ as in lemma tilting-control. Here we fix $C_1 = 50^k$. Notice we use latin indices to mean scale ρ, as opposed to greek indices to mean scale $1/2$. However through this section there shall be no confusion, as *any* scale will be scale ρ.

Further, unless otherwise explicitly specified, any constant c will depend *only* on (n, k, ρ, C_1), and hence only on n.

Here is the key inductive step. We shall devote the entirety of this section to proving it. To reiterate, we write $r_{n_i} = \rho^{n_i}$.

Theorem 5.1. *Take $\{B_{10r_{n_i}}(x_i)\}$ a disjoint collection of balls (the "original" balls), such that $A \geq n_i \geq 1$. Write*

$$\mu_j = \sum_{n_i \geq j} r_{n_i}^k \delta_{x_i}.$$

Suppose for every $j \in \{A, \ldots, 1\}$, and every $x \in B_2$, we have

$$\mu_j B_{r_j}(x) \leq C_1 r_j^k.$$

And suppose also we have, for every $x \in B_2$ and $\alpha_0 \in \{0, 1, 2, \ldots\}$,

$$\sum_{\alpha \geq \alpha_0} \int_{B_{2r_{\alpha_0}}} D_\mu^k(z, 16r_\alpha) d\mu_1(z) \leq r^k \delta^2.$$

Then for $\delta \leq \delta_0(n)$, we have

$$\mu_1 B_1 = \mu_1 B_{r_0}(0) = \sum_{x_i \in B_1} r_{n_i}^k \leq C_1.$$

Remark 5.2. If we apply the above theorem at scale j, to deduce that

$$\mu_{j+1} B_{r_j}(x) \leq C_1 r_j^k$$

for any x, then we've in fact shown

$$\mu_j B_{r_j}(x) \leq C_1 r_j^k,$$

since by disjointness either $\mu_j B_{r_j}(x) = r_j^k$ or $\mu_j B_{r_j}(x) = \mu_{j+1} B_{r_j}(x)$.

Remark 5.3. The integral L^2 closeness condition gives pointwise distortion bounds, by observation d-integral-average. Precisely, for any $x \in B_1$ and any r_i, we have

$$D(x, 8r_i) \leq \frac{c}{r_i^k} \int_{B_{8r_i}(x)} D(z, 16r_i) d\mu(z) \leq c\delta^2.$$

5.4 setup

We construct a sequence of manifolds $T_i' \subset T_i$, which will keep track of volume as we progress down scale. Each T_i will be bi-$W^{1,p}$ close to a disc, and T_i' will be T_i minus a collection of holes. The holes will bound all but a negligable portion of μ-volume.

The two key facts are that on balls of big volume ("good balls"), we have good tilting control (lemma tilting), and negligable μ-volume away from the L^2-best plane. The basic strategy is as follows: cover $T_i \sim$ expanded holes $\subset T_i'$ with balls of scale $i+1$; away from these balls we have μ-negligable volume; divide the $i+1$ balls into "good" and "bad" balls, depending on whether we have big or small volume; cut out small holes for each bad balls, and Reifenberg into each good ball.

We will preserve lower-order volume control by carefully avoiding the original balls at each scale.

To avoid excess notation we will write

$$\mu \equiv \mu_1 = \sum_i r_{n_i}^k \delta_{x_i}.$$

5.5 construction

For each scale i we define a collection of "good balls" $\{B_{r_i}(g_{ij})\}_{j=1}^{G_i}$, and "bad balls" $\{B_{r_i}(b_{ij})\}_{j=1}^{B_i}$, and "final balls" $\{B_{10r_i}(f_{ij})\}_{j=1}^{F_i}$.

Associated to each good ball g_{ij}, we define $L_{ij} = V(g_{ij}, 8r_i)$, and define the "excess set" to be

$$E_{ij} = B_{r_i}(g_{ij}) \sim B_{r_{i+1}/50}(L_{ij}).$$

We let $E_i = \cup_{j=1}^{G_i} E_{ij}$.

Some guidance: μ will have small measure in E_{ij} and bad balls, controlled (and $\sim$small) measure in final balls, and big measure in good balls away from the excess set.

We set

$$R_i = \bigcup_{\ell=0}^{i} \left[\left(\cup_{j=1}^{B_\ell} B_{r_\ell}(b_{\ell j}) \right) \cup \left(\cup_{j=1}^{F_\ell} B_{10r_\ell}(f_{\ell j}) \right) \right].$$

At scale 0, we have B_1 as our only good ball, and no bad or final balls. I.e. $G_0 = 1$, with $g_{01} = 0$, and $B_0 = F_0 = 0$. Consequently $R_0 = \emptyset$.

For $i \geq 1$, the final balls are defined by the conditions

$$\{f_{ij}\}_j = \{x_p : n_p = i \text{ and } x_p \in \cup_j B_{3r_{i-1}/2}(g_{i-1,j})\}.$$

The good and bad balls (for $i \geq 1$) are defined by two conditions. First,

$$\{b_{ij}\}_j \cup \{g_{ij}\}_j$$

form a maximal $2r_i/5$ net in

$$B_1 \cap \left\{ \cup_j \left(B_{r_i/40}(L_{i-1,j}) \cap B_{r_{i-1}}(g_{i-1,j}) \right) \right\} \sim (R_{i-1} \cup \cup_j B_{10r_i}(f_{ij}))$$

Second, the good balls are precisely those for which

$$\mu B_{r_i}(g_{ij}) \geq 32^k \epsilon_k r_i^k.$$

Recall that $L_{ij} = V(g_{ij}, 8r_i)$. Write $p_{ij}^\perp = \mathrm{proj}_{L_{ij}^\perp}$, and $p_{ij} = \mathrm{proj}_{L_{ij}}$, for the associated *linear* projections. Define the map

$$\sigma_i(x) = x - \sum_{j=1}^{G_i} \phi_{ij}(x) p_{ij}^\perp (x - m_{ij}),$$

where a) m_{ij} is the COM associated to $B_{r_i}(g_{ij})$; and b) ϕ_{ij} is a POU such that

$$\begin{aligned}
&\mathrm{spt}\phi_{ij} \subset B_{3r_i}(g_{ij}), \\
&|\nabla \phi_{ij}| \leq 10/r_i, \\
&\sum_j \phi_{ij} = 1 \text{ on } \cup_j B_{5r_i/2}(g_{ij}).
\end{aligned}$$

Now set $T_i = \sigma_i(T_{i-1})$, and let

$$T_i' = \sigma_i(T_{i-1}' \sim \cup_j B_{r_i/5}(f_{ij}) \cup \cup_j B_{r_i/5}(b_{ij})).$$

We let $T_0 = T_0' = L_{0,1} = V(0, 8)$.

5.6 inductive hypothesis

The key properties we need of our manifolds and coverings are the following. We shall prove them inductively.

We need "covering control":

$$B_1 \subset (\cup_j B_{r_i}(g_{ij})) \cup R_i \cup (\cup_{\ell \leq i-1} E_\ell).$$

And "graphicality":

$$(\star_i) \begin{cases} T_i \cap B_{2r_i}(g_{ij}) = \mathrm{graph}_{L_{ij}} f \\ |f|_{C^1_{r_i}} \leq \Lambda(n, k)\delta. \end{cases}$$

And "hole control":

$$T_i' = T_i \text{ inside } B_{10r_{i+1}}(T_i \sim R_i).$$

5.7 covering control

The first property is the easiest.

Lemma 5.8 (covering control)**.** *For every i, we have*

$$B_1 \subset (\cup_j B_{r_i}(g_{ij})) \cup R_i \cup (\cup_{\ell \leq i-1} E_\ell). \tag{2}$$

Proof. Trivially this holds at $i = 0$. Suppose EQREF holds at scale $i - 1$. By construction, for each $j_0 \in \{1, \ldots, G_i\}$ we have

$$\begin{aligned}
B_{r_i/40}(L_{i-1,j_0}) \cap B_{r_{i-1}}(g_{i-1,j_0}) &\subset R_{i-1} \cup B_{r_i}(\{g_{ij}\}_j \cup \{b_{ij}\}_j) \cup B_{10r_i}(\{f_{ij}\}_j) \\
&= \cup_j B_{r_i}(g_{ij}) \cup R_i.
\end{aligned}$$

And therefore

$$\cup_j B_{r_{i-1}}(g_{i-1,j}) \subset \cup_j B_{r_i}(g_{ij}) \cup R_i \cup E_{i-1}.$$

By our inductive hypothesis, we've therefore proven EQREF at scale i. $\square$

5.9 immediate consequences of construction

Before embarking on proving "graphicality" and "hole control", we make some elementary observations about our construction.

A) We have $T'_i = T'_{i-1}$ outside $\cup_j B_{2r_{i-1}}(g_{i-1,j})$. In particular, $T'_i \equiv T_0$ outside B_2.

B) The balls $\{B_{r_i/5}(b_{ij})\}_j \cup \{B_{r_i/5}(f_{ij})\}_j \cup \{B_{r_i/5}(g_{ij})\}_j$ are all disjoint.

C) For any final ball $B_{r_i}(f_{ij})$, with $f_{ij} \in B_{3r_{i-1}/2}(g_{i-1,j_0})$, we in fact have

$$\mathrm{dist}(f_{ij}, L_{i-1,j_0}) \leq c\delta r_{i-1}.$$

This follows immediately from lemma COM-control since f_{ij} is the COM of $B_{r_i}(f_{ij})$.

D) If any one of the original balls $B_{10r_i}(x_p)$ is not a final ball, then $B_{10r_i}(x_p)$ is disjoint from any good $B_{r_i}(g_{ij})$. (this is why, in defining the f_{ij}, we have the $3r_{i-1}/2$ instead of just r_{i-1})

E) Since the excess set is a definite distance from the L^2-best plane, we know this set is very small.

5.10 tilting control in good balls

Here's the name of the game. In any good ball, we need volume control at one lower scale:

Lemma 5.11. *For any $y \in B_{r_i}(g_{ij})$, we have the volume bound*

$$\mu B_{r_{i+1}}(y) \leq C_1 r_{i+1}^k.$$

Proof. By construction (and consequence D) g_{ij} is distance $\geq 10r_i$ from any original r_i ball center. So if $y \in B_{r_i}(g_{ij})$, then y is distance $\geq 9r_i$ from any original r_i ball center. Similarly, for any $\ell \leq i-1$ we know $g_{ij} \in B_{r_\ell}(g_{\ell,j_\ell})$. So g_{ij} is distance $\geq 9r_\ell$ from any original r_ℓ ball center, and hence y is distance $\geq 8r_\ell$ from any original r_ℓ ball center. (since $r_i << r_\ell$).

Therefore $\mu B_{r_{i+1}}(y) = \mu_{i+1} B_{r_{i+1}}(y)$, and the bound follows by the downward-reifenberg-hypothesis. □

This allows us to apply lemma tilting-control between sequential good balls:

Lemma 5.12 (good ball tilting)**.** *Take $g_{ij} \in B_{r_i}(g_{i-1,j_0})$. Then we have*

$$d_H(L_{ij} \cap B_{r_i}(g_{ij}), L_{i-1,j_0} \cap B_{r_i}(g_{ij})) \leq c\delta r_{i-1}$$

In particular, we have

$$d_G(L_{ij}, L_{i-1,j_0}) \leq c\delta, \quad \mathrm{dist}(m_{ij}, L_{i-1,j_0}) \leq c\delta r_{i-1}.$$

and if $|g_{ik} - g_{ij}| < 7$, then

$$d_G(L_{ij}, L_{ik}) \leq c\delta, \quad \mathrm{dist}(m_{ik}, L_{ij}) \leq c\delta r_{i-1}.$$

Proof. We wish to apply lemma tilt-control at scale $B_1 \equiv B_{r_{i-1}}(g_{i-1,j_0})$, with $B_\rho(x) \equiv B_{r_i}(g_{ij})$. The lower volume control is immediate from our construction as good balls. The upper volume control at scale $\rho^2 \equiv r_{i+1}$ follows from the above lemma. By construction we have $\mathrm{dist}(g_{ij}, L_{i-1,j_0}) < r_i/40$. So we satisfy all the required hypothesis of lemma tilt-control, and obtain formula REF. The d_G formulas follow immediately.

To obtain the dist formulas, use lemma COM-control with the facts that $B_{r_i}(g_{ij}) \subset B_{8r_{i-1}}(g_{i-1,j_0})$ and $B_{r_i}(g_{ik}) \subset B_{8r_i}(g_{ij})$. □

In fact we've actually proven the following.

Lemma 5.13 (refined good ball tilting). *Given $g_{ij} \in B_{r_{i-1}}(g_{i-1,j_0})$, we have*

$$d_H(L_{ij} \cap B_{r_i}(g_{ij}), L_{i-1,j_0} \cap B_{r_i}(g_{ij})) \leq c(D(g_{ij}, 8r_i) + D(g_{i-1,j_0}, 8r_{i-1}))^{1/2} r_{i-1}.$$

And consequently

$$d_G(L_{ij}, L_{i-1,j_0}) \leq c(D(g_{ij}, 8r_i) + D(g_{i-1,j_0}, 8r_{i-1}))^{1/2}$$
$$\mathrm{dist}(m_{ij}, L_{i-1,j_0}) \leq cD(g_{i-1,j_0}, 8r_{i-1})^{1/2} r_{i-1},$$

and if $|g_{ik} - g_{ij}| < 7$, then

$$d_G(L_{ij}, L_{ik}) \leq c(D(g_{ij}, 8r_i) + 2D(g_{i-1,j_0}, 8r_{i-1}) + D(g_{ik}, 8r_i))^{1/2}$$
$$\mathrm{dist}(m_{ik}, L_{ij}) \leq cD(g_{ij}, 8r_i)^{1/2} r_i.$$

Good tilting control gives good C^1 bounds on our map σ_i. The coarse estimates are extremely useful but insufficient to prove $(\star_i)$.

Observation 5.14. *In any good ball g_{ij}, we have*

$$D(g_{ij}, 8r_i) \leq \frac{c}{r_i^k} \int_{B_{8r_i}(g_{ij})} D(z, 16r_i) d\mu(z).$$

If $z \in B_{2r_i}(g_{ij})$, then

$$D(z, 16r_{i+1}) \leq cD(z, 16r_i),$$

since $\mu B_{16r_i}(z) \geq \mu B_{r_i}(g_{ij}) \geq 32^k \epsilon_k r_i^k$.

Lemma 5.15 (coarse estimates). *Take $g_{ij} \in B_{r_{i-1}}(g_{i-1,j_0})$. Suppose*

$$\begin{cases} T_{i-1} \cap B_{2r_{i-1}}(g_{i-1,j_0}) = \mathrm{graph}_{L_{i-1,j_0}} f \\ |f|_{C^1_{r_{i-1}}} \leq \beta. \end{cases}$$

Then

$$\sup_{B_{3r_i}(g_{ij}) \cap T_{i-1}} \left(r_i^{-1} |\sigma_i - Id| + |D^\perp_{T_{i-1}}(\sigma_i - Id)| + |D^\top_{T_{i-1}}(\sigma_i - Id)|^{1/2} \right)$$
$$\leq c\beta + c \left(\frac{1}{r_{i-1}^k} \int_{B_{2r_{i-1}}(g_{i-1,j_0})} D(z, 16r_{i-1}) d\mu(z) \right)^{1/2}$$

In particular, if $(\star_{i-1})$ *holds, then*

$$\sup_{T_{i-1}} \left(r_i^{-1}|\sigma_i - Id| + |D_{T_{i-1}}(\sigma_i - Id)| \right) \leq c_c(\Lambda+1)\delta.$$

Proof. Recall that

$$\sigma_i(x) - x = -\sum_j \phi_{ij}(x) p_{ij}^{\perp}(x - m_{ij}),$$

and consequently for any vector V we have

$$D_V(\sigma_i(x) - x) = -\sum_j (D_V \phi_{ij}(x)) p_{ij}^{\perp}(x - m_{ij}) - \sum_j \phi_{ij}(x) p_{ij}^{\perp}(V).$$

Fix a $j \in \{1, \ldots, G_i\}$. By assumption we know

$$B_{3r_i}(g_{ij}) \cap T_{i-1} = \mathrm{graph}_{L_{i-1,j_0}} f \text{ with } |f|_{C^1_{r_{i-1}}} \leq \beta.$$

Now fix an $x \in B_{3r_i}(g_{ij})$, and write $x = (\zeta, f(\zeta))$ for $\zeta \in L_{i-1,j_0}$. Suppose $\phi_{ik}(x) > 0$, so in particular $|g_{ij} - g_{ik}| < 6r_i$. We calculate that

$$\begin{aligned}
|p_{ik}^{\perp}(x - m_{ik})| &\leq |p_{ik}^{\perp} - p_{i-1,j_0}^{\perp}||x - m_{ik}| + |p_{i-1,j_0}^{\perp}(x - \zeta)| + |p_{i-1,j_0}^{\perp}(\zeta - m_{ik})| \\
&\leq c d_G(L_{ik}, L_{i-1,j_0}) r_i + \beta r_{i-1} + \mathrm{dist}(m_{ik}, L_{i-1,j_0}) \\
&\leq c \left(D(g_{ik}, 8r_i) + D(g_{i-1,j_0}, 8r_{i-1}) \right)^{1/2} r_i + c\beta r_i
\end{aligned}$$

Since the balls $\{B_{r_i/5}(g_{ij})\}_j$ are disjoint, there is a $\tilde{C}$, such that the number of k for which $\phi_{ik}(x) > 0$ is $\leq \tilde{C}$. We therefore obtain

$$|\sigma_i(x) - x| \leq c\tilde{C}\beta r_i + c\left(\tilde{C} D(g_{i-1,j_0}, 8r_{i-1}) + \sum_{g_{ik} \in B_{6r_i}(g_{ij})} D(g_{ik}, 8r_i) \right)^{1/2} r_i$$

But by obs good-ball-d-stuff, we have

$$\begin{aligned}
\sum_{g_{ik} \in B_{6r_i}(g_{ij})} D(g_{ik}, 8r_i) &\leq \frac{c}{r_i^k} \int_{\cup_{g_{ik} \in B_{6r_i}(g_{ij})} B_{8r_i}(g_{ik})} D(z, 16r_i) d\mu \\
&\leq \frac{c\tilde{C}}{r_{i-1}^k} \int_{B_{2r_{i-1}}(g_{i-1,j_0})} D(z, 16r_i) d\mu.
\end{aligned}$$

Therefore, again using obs good-ball-d-stuff, we obtain

$$|\sigma_i(x) - x| \leq c\tilde{C}\beta r_i + c\tilde{C} \left(\frac{1}{r_{i-1}^k} \int_{B_{2r_{i-1}}(g_{i-1,j_0})} D(z, 16r_{i-1}) d\mu(z) \right)^{1/2} r_i.$$

The gradient term bound follows precisely as in the original coarse estimates, observing that for any unit tangential W we have

$$\begin{aligned}|W \cdot p_{ik}^{\perp}(Y)| &\leq \left(|p_{ik}^{\perp} - p_{i-1,j_0}^{\perp}| + |p_{i-1,j_0}^{\perp}(W)|\right)|p_{ik}(Y)| \\ &\leq c\,(d_G(L_{ik}, L_{i-1,j_0}) + \beta)\,|p_{ik}(Y)| \\ &\leq c\left[\left(\frac{1}{r_{i-1}^k}\int_{B_{2r_{i-1}}(g_{i-1,j_0})} D(z, 16r_i)d\mu(z)\right)^{1/2} + \beta\right]|p_{ik}(Y)|. \square\end{aligned}$$

Now we can prove $(\star_i)$.

Lemma 5.16 (squash estimates)**.** *Suppose* $(\star_{i-1})$ *holds. Then provided* $\delta \leq \delta_0(n, \Lambda)$*, we have*

$$\begin{cases} T_i \cap B_{2r_i}(g_{ij}) = \operatorname{graph}_{L_{ij}} f \\ |f|_{C^1_{r_i}} \leq c_{sq}\delta. \end{cases}$$

In fact we have the estimate: if $g_{ij} \in B_{r_{i-1}}(g_{i-1,j_0})$*, then*

$$|f|_{C^1_{r_i}} \leq c\left(\frac{1}{r_{i-1}^k}\int_{B_{2r_{i-1}}(g_{i-1,j_0})} D(z, 16r_{i-1})d\mu(z)\right)^{1/2}$$

Proof. Fix a $g_1 = g_{ij}$, $m_1 = m_{ij}$ and $p_1 = p_{ij}$. Then we have

$$\begin{aligned}\sigma_i(x) &= m_1 + p_1(x - m_1) + p_1^{\perp}(x - m_1) - \sum_j \phi_{ij}(x)p_{ij}^{\perp}(x - m_{ij}) \\ &= m_1 + p_1(x - m_1) + e(x).\end{aligned}$$

For $x \in B_{5r_i/2}(g_1)$, since $\sum_j \phi_{ij}(x) = 1$, we have

$$\begin{aligned}e(x) &= \sum_j \phi_{ij}\left(p_1^{\perp}(x - m_1) - p_{ij}^{\perp}(x - m_{ij})\right) \\ &= \sum_j \phi_{ij}\left(p_1^{\perp}(m_{ij} - m_1) + (p_1^{\perp} - p_{ij}^{\perp})(x - m_{ij})\right).\end{aligned}$$

If $\phi_{ik}(x) > 0$, then $|g_1 - g_{ik}| < 6r_i$, and so by lemma tilting-control we have

$$\begin{aligned}|p_1^{\perp}(m_{ik} - m_1)| &\leq \operatorname{dist}(L_1, m_{ik}) + \operatorname{dist}(L_1, m_1) \\ &\leq c\delta r_i.\end{aligned}$$

and

$$\begin{aligned}|(p_1^{\perp} - p_{ik}^{\perp})(x - m_{ik})| &\leq d_G(p_1, p_{ik})|x - m_{ik}| \\ &\leq c\delta r_i.\end{aligned}$$

Since there are at most $\tilde{C}$ values of k for whic $\phi_{ik}(x) > 0$, we obtain $|e(x)| \leq c\delta r_i$.

Similarly, for any unit vector V we have

$$D_V e = \sum_j (D_V \phi_{ij}) \left(p_1^\perp (m_{ij} - m_1) - p_{ij}^\perp (x - m_{ij}) \right) - \sum_j \phi_{ij}(x)(p_1^\perp - p_{ij}^\perp)(V).$$

Using the same estimates, and $|D\phi_{ij}| \leq 10/r_i$, we similarly obtain $|De(x)| \leq c\delta$. We deduce that $|e|_{C^1_{r_i}} \leq c\delta$ on $B_{5r_i/2}(g_1)$.

We wish to apply the squash lemma. Let $g_1 \in B_{r_{i-1}}(g_{i-1,j_0})$ for some j_0, and hence

$$B_{5r_i/2}(g_1) \cap T_{i-1} = \mathrm{graph}_{L_{i-1,j_0}} f, \quad |f|_{C^1_{r_{i-1}}} \leq \Lambda\delta.$$

Since by lemma tilting-control $d_H(L_1 \cap B_{r_i}(g_1), L_{i-1,j_0} \cap B_{r_i}(g_1)) \leq c\delta r_i$, by choosing δ sufficiently small (depending on Λ, n) we can ensure that

$$B_{5r_i/2}(g_1) \cap T_{i-1} = \mathrm{graph}_{L_1} \tilde{f}, \quad |\tilde{f}|_{C^1_{r_i}} \leq 1.$$

Apply the squash lemma at scale $2r_i$ to deduce

$$\sigma_i(B_{5r_i/2}(g_1) \cap T_{i-1}) = \mathrm{graph}_{L_1} f, \quad |f|_{C^1_{r_i}} \leq c\delta.$$

By the coarse estimates we know $|\sigma_i(x) - x| \leq c(1+\Lambda)\delta r_i$. Thus if $c(1+\Lambda)\delta$ is universally small, we can replace $\sigma_i(B_{5r_i/2}(g_1) \cap T_{i-1})$ in the above with $\sigma_i(T_{i-1}) \cap B_{2r_i}(g_1)$. □

By plugging the "refined" squash estimates at scale $i-1$ into the coarse estimates, we obtain.

Lemma 5.17 (refined coarse estimates)**.** *Suppose* $(\star)$ *holds through* $i-1$*. If* $i \geq 2$*, then given* g_{ij}*, we can find a* g_{i-2,j_1}*, so that* $g_{ij} \in B_{r_{i-2}+r_{i-1}}(g_{i-2,j_1})$*, and*

$$\begin{aligned} &\sup_{B_{3r_i}(g_{ij}) \cap T_{i-1}} \left(r_i^{-1}|\sigma_i - Id| + |D^\perp_{T_{i-1}}(\sigma_i - Id)| + |D^\top_{T_{i-1}}(\sigma_i - Id)|^{1/2} \right) \\ &\leq c \left(\frac{1}{r_{i-2}^k} \int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16 r_{i-2}) d\mu(z) \right)^{1/2} \\ &\leq c\delta. \end{aligned}$$

If $i = 1$*, then for any* g_{1j} *we have*

$$\begin{aligned} &\sup_{B_{3r_1}(g_{1j}) \cap T_0} \left(r_1^{-1}|\sigma_1 - Id| + |D^\perp_{T_0}(\sigma_1 - Id)| + |D^\top_{T_0}(\sigma_1 - Id)|^{1/2} \right) \\ &\leq c \left(\int_{B_2} D(z, 16) d\mu \right)^{1/2} \\ &\leq c\delta. \end{aligned}$$

5.18 hole control

Lemma 5.19. *Suppose "graphicality" and "hole-control" both hold at scale* $i-1$*. Then "hole-control" holds at scale* i*.*

Proof. Since we know $T_i' \subset T_i$ by construction, we only need to show one direction. Take $x_0 \in T_i \sim R_i$, and $x \in T_i \cap B_{10r_{i+1}}(x_0)$.

Since our scale $\rho \leq 1/10$, we have $d(x, \{b_{ij}\}_j \cup \{f_{ij}\}_j) \geq 4r_i/5$. Then by the coarse estimates, $d(\sigma_i^{-1}(x), \{b_{ij}\} \cup \{f_{ij}\}) \geq 3r_i/5$. So

$$\sigma_i^{-1}(x) \in T_{i-1} \sim B_{r_i/5}(\{f_{ij}\}_j \cup \{b_{ij}\}_j).$$

By our inductive hypothesis "hole control", it will suffice to show that $d(\sigma_i^{-1}(x), T_{i-1} \sim R_{i-1}) < 10r_i$. This follows trivially from the coarse estimates, since $\sigma_i^{-1}(x_0) \in \sigma_i^{-1}(T_i \sim R_i)$. □

Now combine all the above lemmas to deduce!

Theorem 5.20. *The inductive properties "covering control", "graph control", and "hole control" hold at every scale* $i \geq 0$*.*

Proof. Take $\Lambda = c_{sq}$, the constant from the squash estimates. Now choose δ sufficiently small, depending only on n, so that all the arguments work. □

5.21 cutting out holes

Lemma 5.22. *We have*

$$\begin{aligned} \omega_k(B_i + F_i)(r_i/10)^k &\leq \mathcal{H}^k(\textit{holes we cut out from } T_{i-1}') \\ &= \mathcal{H}^k(T_{i-1}') - \mathcal{H}^k(\sigma_i^{-1}(T_i')) \end{aligned}$$

Proof. Let $b_{ij} \in B_{r_{i-1}}(g_{i-1,j_0})$. We have $\text{dist}(b_{ij}, L_{i-1,j_0}) < r_i/40$, and hence by graphicality we know

$$\text{dist}(b_{ij}, T_{i-1}) \leq \Lambda\delta r_{i-1} + r_i/40 < r_i/30 \tag{3}$$

for δ sufficiently small (depending on Λ, ρ).

I claim that in $B_{r_i/5}(b_{ij})$ we know T_{i-1}' is C^1-close to a disc. Together these facts imply:

$$\mathcal{H}^k(T_{i-1}' \cap B_{r_i/5}(b_{ij})) \geq \omega_k(r_i/10)^k.$$

Let's prove my claim. Since $b_{ij} \notin R_{i-1}$, we trivially $B_{r_i/5}(b_{ij}) \subset B_{10r_i}(T_{i-1} \sim R_{i-1})$. So $T_{i-1} = T_{i-1}'$ in $B_{r_i/5}(b_{ij})$. And of course $B_{r_i/5}(b_{ij}) \subset B_{2r_{i-1}}(g_{i-1,j_0})$, so we have graphical control of T_{i-1} in this region. This proves the claim.

Now pick the j_0 so that $f_{ij} \in B_{3r_{i-1}/2}(g_{i-1,j_0})$. Then consequence C) says

$$\text{dist}(f_{ij}, L_{i-1,j_0}) \leq c\delta r_{i-1}.$$

Again we know that $B_{r_i/5}(f_{ij}) \subset B_{2r_{i-1}}(g_{i-1,j_0})$, and in the relavent region $T'_{i-1} = T_{i-1}$ with good graphical control. Therefore, provided δ is sufficiently small, we have as before an estimate like

$$\mathcal{H}^k(T'_{i-1} \cap B_{r_i/5}(f_{ij})) \geq \omega_k (r_i/10)^k.$$

Since we know all these holes are disjoint (consequence B), the inequality follows immediately. $\square$

5.23 manifold volume control

Lemma 5.24. *Suppose* $(\star)$ *holds through* $i-1$. *Take* $\Omega \subset T_{i-1}$. *Then for* $i \geq 2$ *we have*

$$\mathcal{H}^k(\sigma_i(\Omega)) \leq \mathcal{H}^k(\Omega) + c\int_{B_2} D(z, 16r_{i-2}) d\mu(z).$$

If $i = 1$, *then we have*

$$\mathcal{H}^k(\sigma_1(\Omega)) \leq \mathcal{H}^k(\Omega) + c\int_{B_2} D(z, 16) d\mu(z).$$

Proof. Take $\Omega \subset T_{i-1}$. We know $\sigma_i = Id$ outside $\cup_j B_{3r_i}(g_{ij})$, so

$$\mathcal{H}^k(\sigma_i(\Omega \sim \cup_j B_{3r_i}(g_{ij}))) = \mathcal{H}^k(\Omega \sim \cup_j B_{3r_i}(g_{ij})).$$

We estimate volume within the good $3r_i$ balls, first assuming $i \geq 2$. Given g_{ij}, we know

$$\mathcal{H}^k(\Omega \cap B_{3r_i}(g_{ij}) \leq (1 + c\delta) r_i^k$$

by $(\star_{i-1})$. We also know from the refined coarse estimates that the Jacobian is bounded like

$$J\left(\sigma_i|_{B_{3r_i}(g_{ij})}\right) \leq 1 + \frac{c}{r_{i-2}^k}\int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16r_{i-2}) d\mu.$$

We elaborate. For notation simplicity write the integral error above as δ^2, so the RHS is $1 + \delta^2$. We have $D_{T_{i-1}}\sigma_i = Id + D_{T_{i-1}}(\sigma_i - Id)$.

Take $x \in B_{3r_i}(g_{ij})$, and identify $T_x T_{i-1}$ with $R^k \times \{0\}$. We then have by the improved coarse estimates that

$$D_{T_{i-1}}\sigma_i(x) = \left[\frac{Id + O(\delta^2)}{O(\delta)}\right].$$

This then gives us

$$(D_{T_{i-1}}\sigma_i(x))^t D_{T_{i-1}}\sigma_i(x) = Id_{k\times k} + O(\delta^2)$$

and hence $J\sigma_i = \sqrt{1 + O(\delta^2)} = 1 + O(\delta^2)$.

So for any $P \subset B_{3r_i}(g_{ij})$, we have

$$\begin{aligned}\mathcal{H}^k(\sigma_i(\Omega \cap P)) &\leq \mathcal{H}^k(\Omega \cap P)\left(1 + \frac{c}{r_{i-2}^k}\int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16r_{i-2})d\mu\right)\\ &\leq \mathcal{H}^k(\Omega \cap P) + c\int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16r_{i-2})d\mu.\end{aligned}$$

Choose $\{P_j\}_{j=1}^{G_i}$ a measurable partition of $\cup_j B_{3r_i}(g_{ij})$, so that $P_j \subset B_{3r_i}(g_{ij})$. There is $\tilde{C} = \tilde{C}(\rho, n)$ for that: a) the number of good r_i balls in any fixed r_{i-2} good ball is $\leq \tilde{C}$; and b) at most $\tilde{C}$ good r_{i-2} balls overlap at any point. We thus have

$$\begin{aligned}&\mathcal{H}^k(\sigma_i(\Omega \cap \cup_j B_{3r_i}(g_{ij}))) - \mathcal{H}^k(\Omega \cap \cup_j B_{3r_i}(g_{ij}))\\ &= \sum_j \mathcal{H}^k(\sigma_i(\Omega \cap P_j)) - \mathcal{H}^k(\Omega \cap P_j)\\ &\leq \tilde{C}c\int_{\cup_j B_{2r_{i-2}}(g_{i-2,j})} D(z, 16r_{i-2})d\mu\\ &\leq \tilde{C}^2 c\int_{B_2} D(z, 16r_{i-2})d\mu.\end{aligned}$$

This proves the estimates when $i \geq 2$. For $i = 1$ we use the same proof, except it is slightly simplified in that we only have a single good $r_0 = 1$ ball. □

5.25 bounding μ-mass

Observation 5.26. *For each i we have by "covering control"*

$$\mu B_1 \leq \sum_{j=1}^{G_i} \mu B_{r_i}(g_{ij}) + \sum_{\ell \leq i}\left(\sum_{j=1}^{B_\ell} \mu B_{r_\ell}(b_{\ell j}) + \sum_{j=1}^{F_\ell} B_{10r_\ell}(f_{\ell j}) + \mu E_\ell\right).$$

Lemma 5.27. *We have*

$$\mu E_i \leq c\int_{B_2} D(z, 16r_i)d\mu(z).$$

Proof. For each j, by definition of the L^2-best plane we have

$$\begin{aligned}\mu E_{ij} &\leq \frac{c}{r_i^2}\int_{B_{r_i}(g_{ij})} \operatorname{dist}(z, L_{ij})^2 d\mu(z)\\ &= cr_i^k D(g_{ij}, 8r_i)\\ &\leq cr_i^k ⨍_{B_{8r_i}(g_{ij})} D(z, 16r_i)d\mu(z)\\ &\leq c\int_{B_{8r_i}(g_{ij})} D(z, 16r_i)d\mu(z).\end{aligned}$$

Therefore, since at most $\tilde{C}$ of the $\{B_{8r_i}(g_{ij})\}_j$ overlap at any point, the bound follows in the obvious way. $\square$

Lemma 5.28. *For every i, we have*

$$\sum_{\ell \le i} \omega_k (B_\ell + F_\ell)(r_\ell/10)^k \le \mathcal{H}^k T_0 - \mathcal{H}^k T_i' + c \sum_{\ell=0}^{\max(i-2,0)} \int_{B_2} D(z, 16r_\ell) d\mu(z).$$

Proof. Apply lemma manifold-volume-control to the set $\sigma_i^{-1}(T_i' \cap B_2)$, then use lemma cutting-out-holes, to obtain

$$\begin{aligned} \mathcal{H}^k T_i' - \mathcal{H}^k T_{i-1}' &\le \mathcal{H}^k \sigma_i^{-1} T_i' - \mathcal{H}^k T_{i-1}' + c \int_{B_2} D(z, 16r_{i-2}) d\mu \\ &\le -\omega_k (B_i + F_i)(r_i/10)^k + c \int_{B_2} D(z, 16r_{i-2}) d\mu. \end{aligned}$$

Now iterate this relation. $\square$

Theorem 5.29. *For any i, we have*

$$10^{-k} \mu B_1 \le \sum_{j=1}^{G_i} 10^{-k} \mu B_{r_i}(g_{ij}) + c \sum_{\ell \le i} \int_{B_2} D(z, 16r_\ell) d\mu(z) + 2^k$$

Proof. Note that our disjointness hypothesis, for any final ball f_{ij} we have $\mu B_{10r_i}(f_{ij}) = r_i^k$. Further, since $32^k \epsilon_k \le 1$, we have for any bad ball b_{ij} that $\mu B_{r_i}(b_{ij}) \le r_i^k$.

We use the previous two lemmas and observation:

$$\begin{aligned} 10^{-k} \mu B_1 - \sum_{j=1}^{G_i} 10^{-k} \mu B_{r_i}(g_{ij}) &\le \sum_{\ell \le i} \left((F_\ell + B_\ell)(r_\ell/10)^k + \mu E_\ell \right) \\ &\le c \sum_{\ell \le i} \int_{B_2} D(z, 16r_\ell) d\mu + \omega_k^{-1} (\mathcal{H}^k T_0 - \mathcal{H}^k T_i') \end{aligned}$$

But by consequence A) we have $\mathcal{H}^k T_0 - \mathcal{H}^k T_i' \le \mathcal{H}^k (T_0 \cap B_2)$. $\square$

5.30 finishing the proof

At each scale we have know any good ball $B_{r_i}(g_{ij})$ is at distance $\ge 8r_i$ from any original ball r_ℓ-center, for $\ell \le i$. Therefore if $i = A$, every would-be good ball $B_{r_i}(g_{ij})$ is actually disjoint from every original ball center, and hence has μ measure 0. So at scale $i = A$ we don't have any good balls, and are left with the estimate

$$\begin{aligned} 10^{-k} \mu B_1 &\le c \sum_{\ell \le A} \int_{B_2} D(z, 16r_\ell) d\mu + \omega_k 2^k \\ &\le c\delta + 2^k \\ &\le 5^k \end{aligned}$$

provided δ is sufficiently small, depending only on n. We therefore have the estimate

$$\mu B_1 \leq 50^k = C_1.$$

6 proving discrete Reifenberg

To prove the actual discrete Reifenberg theorem, we inductively use the downwards-discrete-Reifenberg theorem to obtain volume bounds at bigger scales. Fix $\rho = \rho_0$ as in the downwards Reifenberg section.

Take $\{B_{r_p}(x_p)\}$ the disjoint balls hypothesized in theorem discrete-Reifenberg, and $\mu = \sum_p r_p^k \delta_{x_p}$ the associated measure. As the hypotheses and conclusion depend only on n, we can suppose the number of balls is finite. For each p, let n_p be defined by the condition

$$\rho^{n_p} \leq r_p/10 < \rho^{n_p - 1}.$$

We now have a finite, disjoint collection of balls $\{B_{10r_{n_p}}(x_p)\}$, where as in the previous section $r_{n_p} \equiv \rho^{n_p}$. Let $A = \max_p n_p$, and define

$$\mu_j = \sum_{n_p \geq j} r_{n_p}^k \delta_{x_p}.$$

Now since $\mu_j \leq \mu$, we trivially have $D_{\mu_j}(x, r) \leq D_\mu(x, r)$ for any $B_r(x)$. And hence, for any $x \in B_2$ and $r \in (0, 1)$, we also have

$$\sum_{r_\alpha \leq r} \int_{B_{2r}(x)} D_{\mu_j}(z, 16r_\alpha) d\mu_j(z) \leq \sum_{r_\alpha \leq r} \int_{B_{2r}(x)} D_\mu(z, 16r_\alpha) d\mu(z) \leq r^k \delta^2.$$

I claim that, for every $j \geq 0$, $x \in B_1$, we have

$$(\dagger_j) \quad \mu_j B_{r_j}(x) \leq C_1 r_j^k,$$

with $C_1 = 50^k$ as in section "downwards Reifenberg".

First, let's see how this implies the theorem. Taking $x = 0$, and and $j = 0$, the claim at scale 0 implies

$$\begin{aligned} C_1 &\geq \mu_0 B_1 \\ &= \sum_{x_p \in B_1} (\rho^{n_p})^k \\ &\geq \sum_{x_p \in B_1} (\rho/10)^k r_p^k. \end{aligned}$$

And therefore discrete Reifenberg holds with $(10/\rho)^k C_1 \leq C_{dr} = C_{dr}(n)$.

We now prove the claim. Vacuously, $(\dagger)$ holds at scales $j < A$. Now suppose $(\dagger)$ holds up through scale $j + 1$. Pick any x and r_j. Observe that, at scale

$B_1 \equiv B_{r_j}(x)$, all the hypothesis of theorem downwards-discrete-reifenberg are satisifed. Therefore we can deduce that

$$\mu_{j+1} B_{r_j}(x) \leq C_1 r_j^k.$$

But as in remark move-j-up, this now proves $(\dagger_j)$, since by disjointnes we have either $\mu_j B_{r_j}(x) = r_j^k$, or $\mu_j B_{r_j}(x) = \mu_{j+1} B_{r_j}(x)$. This proves the claim, and hence theorem discrete Reifenberg.

7 rectifiable Reifenberg

A similar scheme can be used to prove finite k-rectifiability of a set S, provided the measure $\mu_S = \mathcal{H}^k \llcorner S$ satisfies the same distortion bounds of theorem discrete-reifenberg. In fact one can prove a kind of generalized uniform rectifiability, of David-Semmes, in the sense that on any ball with sufficiently large mass, a fixed percentage coincides with a Lipschitz graph. For ease of notation write $D^k_S \equiv D^k_{\mu_S}$.

Theorem 7.1. *Suppose $S \subset R^n$ satisfies: for any $x \in B_2$, and any $\alpha_0 \in \{0, 1, \ldots\}$, we have*

$$\sum_{\alpha \geq \alpha_0} \int_{B_{2r_{\alpha_0}}(x)} D^k_S(z, 32 r_\alpha) d\mu_S(z) \leq r^k \delta^2.$$

Then for $\delta \leq \delta_0(n)$, S is rectifiable, and has the uniform upper density bounds:

$$\mathcal{H}^k(S \cap B_r(x)) \leq C_{rr}(n) r^k$$

for every $x \in B_1$, $r \in (0, 1/2)$.

Here is the strategy, vaguely. Suppose μ satisfies a condition like in discrete-Reifenberg. Suppose $\mu B_1 \geq \epsilon_k$. Construct a sequence of manifolds $T_i' \subset T_i$ as before, and prove inductively that

$$S \cap B_1 \subset B_{cr_i}(T_i) + \text{small error}.$$

This is almost identical to the discrete-reifenberg construction, and even easier because we have no 'final balls'.

Define the maps $\tau_i = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_1$ as in the original Reifenberg's disc theorem, and recall that $T_i = \tau_i(T_0)$. As before the τ_i limit in C^0 to a map τ_∞. Writing $T_\infty = \tau_\infty(T_0)$, we have that

$$\mu(B_1 \cap T_\infty) \geq \mu B_1 - \text{small error} \geq \frac{1}{2} \mu B_1.$$

The new ingredient is demonstrating uniform $W^{1,p}$-bounds on the maps $\tau_i : T_0 \to R^n$. p can be anything $\in [1, \infty)$, but δ depends on p. The bound passes to the limit, and we get that $\tau_\infty \in W^{1,p}$ also. Provided $p > k$, then most of the image of τ_∞ is a Lipschitz graph.

Since for μ-a.e. x we have $\Theta^{*,k}(\mu, x) \in [2^{-k}, 1]$ (see the next section), we can always choose a scale on which we have large volume. Combining this with the above (at scale $B_1 \to B_{r_x}(x)$) will show S is rectifiable.

7.2 obtaining volume bounds

We first show how to obtian uniform upper density bounds. Actually, as far as Naber's result goes, we don't need to worry about upper volume control, since he first proves uniform volume control, and then applies rectifiable reifenberg. But it's a nice result on it's own.

Theorem 7.3. *Suppose S satisfies: for any $x \in B_2$, and any $\alpha_0 \in \{0, 1, \ldots\}$, we have*

$$\sum_{\alpha \geq \alpha_0} \int_{B_{2r_{\alpha_0}}(x)} D_S^k(z, 32r_\alpha) d\mu_S(z) \leq r^k \delta^2.$$

Then for $\delta \leq \delta_1(n)$, we have for any $x \in B_1$ and any $r \in (0, 1/2)$,

$$\mathcal{H}^k(S \cap B_r(x)) \leq C_{rr}(n) r^k.$$

Proof. Take S and μ_S as above, with μ_S satisfying the distortion hypothesis. We use the discrete Reifenberg theorem to obtain uniform local volume bounds by "discretizing" S. Here's how we do it.

First, observe that S is necessarily σ-finite. This holds because our distortion hypothesis implies

$$\int_{B_2} D(z, 1) d\mu(z) \leq \delta^2.$$

Each set $S_a = S \cap B_2 \cap \{D(z, 1) \geq a > 0\}$ has finite $\mathcal{H}^k$-measure. The set $S_0 = S \cap B_2 \cap \{D(z, 1) = 0\}$ also has finite $\mathcal{H}^k$-measure because, away from a set of measure 0, S_0 is contained in some k-dimensional space. So S is σ-finite.

Therefore it suffices to prove the theorem for any $\mathcal{H}^k$-finite subset of S, as the result for S will follow by considering a countable, $\mathcal{H}^k$-finite exhaustion. Now given any $S' \subset S$, the associated measure $\mu_{S'} = \mathcal{H}^k \llcorner S' \leq \mu_S$, and so the same distoration hypothesis holds for $\mu_{S'}$. Hence there is no loss in generality assuming that $\mathcal{H}^k(S) < \infty$.

Therefore, since μ_S is Radon, we know that $\mathcal{H}^k$-a.e. $x \in S$, we have

$$\Theta^{*,k}(\mu_S, x) \in [2^{-k}, 1], \tag{4}$$

so for μ_S-a.e. x we have a good scale $r_x \leq 1$ for which

$$\mathcal{H}^k(B_{r_x/25}(x) \cap S) \geq 2^{-k-1} \omega_k (r_x/25)^k$$
$$\mathcal{H}^k(B_r(x) \cap S) \leq 2\omega_k r^k \quad \forall r \leq r_x.$$

Of course we can replace S with the set S^* of points x satisfying condition EQREF. Obtain a Vitali sub-covering $\{B_{r_{x_i}}(x_i)\}$ of $\{B_{r_x}(x)\}_{x \in S^*}$. So $\{B_{r_{x_i}/5}(x_i)\}$ are disjoint. Define the measure

$$\mu = \sigma_k 10^{-k} \sum_i r_{x_i}^k \delta_{m_i}$$

where m_i is the center of mass associated to $B_{r_{x_i}/25}(x_i)$, and σ_k is a constant to be determined. Since $m_i \in B_{r_{x_i}/25}(x_i)$, we have that the balls $\{B_{r_{x_i}/10}(m_i)\}$ are disjoint.

I claim that μ satisfies the distoration hypothesis of theorem discrete-reifenberg, for an appropriately small choice of σ_k. Take an arbitrary $B_r(x)$. If for some i we have $B_{r_{x_i}/10}(m_i) \supset B_r(x)$, then trivially $D_\mu(x,r) = 0$, since the support in $B_r(x)$ is only a single point.

So WLOG we can suppose that $r_{x_i}/10 < 2r$ for every $x_i \in B_r(x)$. If $\mu B_r(x) \geq \epsilon_k r^k$, then

$$\begin{aligned}
\mu_S B_{2r}(x) &\geq \sum_{m_i \in B_r(x)} \mu_S B_{r_{x_i}/25}(x_i) \\
&\geq 2^{-k-1} 25^{-k} \omega_k \sum_{m_i \in B_r(x)} r_{x_i}^k \\
&\geq \frac{2^{-2k-1} \cdot 10^k \cdot 25^{-k} \omega_k}{\sigma_k} \epsilon_k (2r)^k.
\end{aligned}$$

Set $\sigma_k = 2^{-2k-1} \cdot 10^k \cdot 25^{-k} \omega_k$. We now calculate

$$\begin{aligned}
r^{k+2} D_\mu(x,r) &= 10^{-k} \sigma_k \inf_L \sum_{m_i \in B_r(x)} d(m_i, L)^2 r_{x_i}^k \\
&\leq c(k) \inf_L \sum_{m_i \in B_r(x)} r_{x_i}^k \fint_{B_{r_{x_i}/25}(x_i)} d(z,L)^2 d\mu_S \\
&\leq c(k) \inf_L \sum_{m_i \in B_r(x)} \int_{B_{r_{x_i}/25}(x_i)} d(z,L)^2 d\mu_S \\
&\leq c(k) \inf_L \int_{B_{2r}(x)} d(z,L)^2 d\mu_S \\
&= c(k) r^{k+2} D_S(x, 2r).
\end{aligned}$$

So the hypothesis of theorem volume-bounds-reifeneberg implies directly the distortion bounds of theorem discrete-reifenberg. Applying discrete-reifenberg at scale $B_1 \equiv B_r(x)$, we obtain

$$\begin{aligned}
C_{dr}(n) r^k &\geq \mu B_r(x) \\
&= 10^{-k} \sigma_k \sum_{x_i \in B_r(x)} r_{x_i}^k \\
&\geq \frac{10^{-k}\sigma_k}{2\omega_k} \sum_{x_i \in B_r(x)} \mu_S(B_{r_{x_i}}(x_i)) \\
&\geq \frac{10^{-k}\sigma_k}{2\omega_k} \mu_S(B_r(x)).
\end{aligned}$$

Simply take $\frac{2 \cdot 10^k \omega_k}{\sigma_k} C_{dr} \leq C_{rr} = C_{rr}(n)$. □

7.4 downwards rectifiable reifenberg

We will fix our scale $\rho = 2^{-\beta} \leq \min\{\rho_0(C_{rr}), 1/10\}$, with ρ_0 as in lemma tilting-control and C_{rr} as in theorem uniform-mass-bounds. As before set $r_i = \rho^i$.

Here is the key step. It shows that in any ball of big volume we are mostly rectifiable.

Theorem 7.5. *Suppose $S \subset R^n$ is such that $\mathcal{H}^k(S \cap B_1) \geq 2^{-k-1}\omega_k$, and satisfies: for any $x \in B_2$, and any $\alpha_0 \in \{0, 1, \ldots\}$, we have*

$$\sum_{\alpha \geq \alpha_0} \int_{B_{2r_{\alpha_0}}(x)} D^k_S(z, 32r_\alpha) d\mu_S(z) \leq r^k \delta^2.$$

Then for $\delta \leq \delta_2(p, n)$, we have a $W^{1,p}$ map $\phi : R^k \to R^n$ so that

$$\mathcal{H}^k(S \cap \phi(B^k_1) \cap B_1) \geq \frac{1}{2}\mathcal{H}^k(S \cap B_1).$$

Remark 7.6. By the first theorem volume-bounds we have also uniform upper density bounds. In fact, by using that a $W^{1,p}$-map is a-Lipschitz away from a small set (of measure $\sim 1/a$) one can prove that

$$\mathcal{H}^k(S \cap B_1) \leq (1 + \epsilon)\omega_k,$$

taking δ sufficiently small depending on n, p, ϵ.

As in discrete-reifenberg we construct a sequence of approximating manifolds, as images $T_i = \sigma_i(T_{i-1}) = (\sigma_i \circ \cdots \circ \sigma_1)(T_0)$. The map ϕ will be the limit of these compositions.

Here is the construction. At each scale we have the usual "good balls" $\{B_{r_i}(g_{ij})\}_{j=1}^{G_i}$ and "bad balls" $\{B_{r_i}(b_{ij})\}_{j=1}^{B_i}$, defined in the exact same way. For each good ball we have $L_{ij} = V(g_{ij}, 8r_i)$, and

$$E_{ij} = B_{r_i}(g_{ij}) \sim B_{r_{i+1}/50}(L_{ij}).$$

Let $E_i = \cup_{j=1}^{G_i} E_{ij}$.

The remainder set at scale i is defined by $R_i = \cup_{\ell=0}^{i} \cup_{j=1}^{B_\ell} B_{r_i}(b_{ij})$.

At scale 0, we have B_1 as our only good ball, and no bad balls. For $i \geq 1$, we defined $\{g_{ij}\}_j \cup \{b_{ij}\}_j$ to be a maximal $2r_i/5$-net in

$$B_1 \cap \left(\cup_j (B_{r_i/40}(L_{i-1,j}) \cap B_{r_{i-1}}(g_{i-1,j}))\right) \sim R_{i-1}.$$

The good balls are those with $\mu B_{r_i}(g_{ij}) \geq 32^k r_i^k$.

Define σ_i, and the POU ϕ_{ij}, in precisely the same way. Set $T_i = \sigma_i(T_{i-1})$, and

$$T'_i = \sigma_i(T'_{i-1} \sim \cup_j B_{r_i/5}(b_{ij})).$$

We initialize with $T_0 = T'_0 = L_{01} = V(0, 8)$.

We prove the following inductive hypotheses, for each scale i. Trivially all three are satisfied at scale 0.

We require "manifold covering control":

$$B_1 \subset B_{r_{i+1}}(T_i) \cup R_i \cup (\cup_{\ell \leq i} E_\ell)\,.$$

And "graphicality":

$$\begin{cases} T_i \cap B_{2r_i}(g_{ij}) = \mathrm{graph}_{L_{ij}} f \\ |f|_{C^1_{r_i}} \leq \Lambda(n,k)\delta. \end{cases}$$

And "hole control":

$$T_i' = T_i \text{ inside } B_{10r_{i+1}}(T_i \sim R_i).$$

Lemma 7.7. *"graphicality" and "hole control" hold at every scale. In fact the coarse, squash, and refined coarse estimates hold for every scale.*

Proof. Suppose both conditions hold at scale $i-1$. By theorem mass-bounds, lemma tilting-control-in-good-balls holds with C_{rr} in place of C_1. Therefore we have tilting control between good balls, and the coarse, squash, and refined coarse estimates all hold at scale i. Therefore graphicality holds for i. The proof of "hole control" is precisely the same as in discrete reifenberg. □

Lemma 7.8. *"manifold covering control" holds at every scale.*

Proof. From "covering control" we have

$$B_1 \subset (\cup_j B_{r_i}(g_{ij})) \cup R_i \cup (\cup_{\ell \leq i-1} E_\ell).$$

In any good r_i ball, we have by "graphicality" that

$$B_{r_i}(g_{ij}) \subset B_{r_{i+1}/40}(L_{ij}) \cup E_{ij} \subset B_{r_{i+1}}(T_i) \cup E_{ij}.$$

And the lemma follows immediately. □

Remark 7.9. There is almost certainly an inclusion like

$$B_1 \subset B_{Cr_i}(T_i') \cup R_i \cup (\cup_{\ell \leq i} E_\ell).$$

7.10 $W^{1,p}$ estimates

We prove a uniform $W^{1,p}$-bound on the approximating maps $\tau_\ell = \sigma_\ell \circ \sigma_{\ell-1} \circ \cdots \circ \sigma_1$. Notice that we do not prove the sequence $\{\tau_\ell\}$ is $W^{1,p}$-Cauchy, only that it is bounded. By weak-$*$ compactness this suffices to give a $W^{1,p}$ bound on the limit.

For a linear transformation A we will write $|A|$ for the operator norm, i.e. $|A| = \sup_{|v| \leq 1} |A(v)|$. For ease of notation write $\tau_{\ell,i} = \sigma_\ell \circ \cdots \circ \sigma_i$.

Lemma 7.11. *Suppose*

$$\fint_{B_{5r_i}(x)\cap T_i} |D_{T_i}(\sigma_\ell \circ \cdots \circ \sigma_{i+1})|^p d\mathcal{H}^k \leq M$$

for any $x \in T_i$. *Then for any* $\Omega \subset T_{i-1}$, *we have*

$$\begin{aligned}
&\int_\Omega |D_{T_{i-1}}(\sigma_\ell \circ \cdots \circ \sigma_i)|^p d\mathcal{H}^k \\
&\quad \leq \int_{\sigma_i(\Omega)} |D_{T_i}(\sigma_\ell \circ \cdots \sigma_{i+1})|^p d\mathcal{H}^k + cMp \int_{B_{5r_{i-2}}(\Omega)} D(z, 16r_{i-2}) d\mu(z).
\end{aligned}$$

If $i = 1$, *then the integrand on the RHS is* $D(z, 16)$.

Proof. We have by the refined estimates that

$$|D_{T_{i-1}}\sigma_i| \leq 1 + \frac{c}{r_{i-2}^k} \int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16r_{i-2}) d\mu(z).$$

The bound follows because of the power-improvement of tangential derivatives.

Therefore for any $P \subset B_{3r_i}(g_{ij})$, we have for some j_1

$$\begin{aligned}
&\int_{\Omega\cap P} |D_{T_{i-1}}\tau_{\ell,i}|^p d\mathcal{H}^k \\
&\quad \leq \int_{\sigma_i(\Omega\cap P)} |D_{T_i}\tau_{\ell,i+1}|^p |D_{T_{i-1}}\sigma_i|^p J(\sigma_i^{-1}) d\mathcal{H}^k \\
&\quad \leq \int_{\sigma_i(\Omega\cap P)} |D_{T_i}\tau_{\ell,i+1}|^p d\mathcal{H}^k \left(1 + \frac{pc}{r_{i-2}^k} \int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16r_{i-2}) d\mu(z)\right) \\
&\quad \leq \int_{\sigma_i(\Omega\cap P)} |D_{T_i}\tau_{\ell,i+1}|^p d\mathcal{H}^k + pMc \int_{B_{2r_{i-2}}(g_{i-2,j_1})} D(z, 16r_{i-2}) d\mu(z).
\end{aligned}$$

We used our hypothesis with $B_{5r_i}(x) \supset \sigma_i(\Omega \cap P)$, and the fact that at scales B_{2r_i} T_i is C^1 close to a plane.

Let $\{P_j\}_{j=1}^{G_i}$ be a measurable partition of $\cup_j B_{3r_i}(g_{ij})$, so that $P_j \subset B_{3r_i}(g_{ij})$. Choose $\tilde{C}$ so that a) at most $\tilde{C}$ good r_i balls fit inside any good r_{i-2} ball; and b) at most $\tilde{C}$ good r_{i-2} balls overlap. Recalling that $\sigma_i \equiv Id$ outside $\cup_j B_{3r_i}(g_{ij})$, we have

$$\begin{aligned}
&\int_\Omega |D_{T_{i-1}}\tau_{\ell,i}|^p - \int_{\sigma_i(\Omega)} |D_{T_i}\tau_{\ell,i+1}|^p \\
&\quad = \sum_j \int_{\Omega\cap P_j} |D_{T_{i-1}}\tau_{\ell,i}|^p - \int_{\sigma_i(\Omega\cap P_j)} |D_{T_i}\tau_{\ell,i+1}|^p \\
&\quad \leq Mp\tilde{C}c \int_{\Omega\cap\cup_j B_{2r_{i-2}(g_{i-2,j})}} D(z, 16r_{i-2}) d\mu(z) \\
&\quad \leq Mp\tilde{C}^2 c \int_{B_{5r_{i-2}}(\Omega)} D(z, 16r_{i-2}) d\mu(z).
\end{aligned}$$

□

Lemma 7.12. *Provided $\delta \leq \delta_1(n,p)$, we have for each ℓ*

$$\int_{T_0} |D_{T_0}(\sigma_\ell \circ \cdots \circ \sigma_1)|^p \leq C_1 \omega_k 5^k$$

Proof. Fix ℓ. We prove inductively that on i that

$$(\dagger_i) \quad \fint_{B_{5r_{i-1}(x)} \cap T_{i-1}} |D_{T_{i-1}}(\sigma_\ell \circ \ldots \circ \sigma_i)|^p d\mathcal{H}^k \leq C_1, \quad \text{for any } x \in T_{i-1}.$$

Clearly when $i = 1$ the theorem is established.

For $\ell = i$ this follows immediately from the refined coarse estimates:

$$\fint_{\Omega \cap T_{i-1}} |D_{T_{i-1}} \sigma_i|^p \leq 1 + pc\delta \leq C_1.$$

for any domain Ω.

Suppose $(\dagger)$ holds for integers $i, \ldots, \ell$. We with to iterate lemma REF. For any domain $\Omega \subset T_{i-1}$ we have $B_{5r_{i-1}}(\sigma_i(\Omega)) \subset B_{5r_{i-2}}(\Omega)$ by the coarse estimates. Taking $\Omega = B_{5r_{i-1}}(x) \cap T_{i-1}$ where $x \in T_{i-1}$, we have

$$\begin{aligned}
\int_{B_{5r_{i-1}}(x) \cap T_{i-1}} |D_{T_{i-1}} \tau_{\ell,i}|^p &\leq \int_{\tau_{\ell-1,i}(\Omega)} |D_{T_{\ell-1}} \sigma_\ell|^p + cC_1 p \sum_{j=0}^{\ell} \int_{B_{5r_{i-2}}(\Omega)} D(z, 16r_j) d\mu(z) \\
&\leq (1 + pc\delta)\mathcal{H}^k(\Omega) + cp \sum_{j=0}^{\ell} \int_{B_{10r_{i-2}}(x)} D(z, 16r_j) d\mu(z) \\
&\leq (1 + pc\delta)\mathcal{H}^k(B_{5r_{i-1}} \cap T_{i-1}) + cp\delta^2 (10r_{i-2})^k \\
&\leq C_1 \mathcal{H}^k(B_{5r_{i-1}}(x) \cap T_{i-1}),
\end{aligned}$$

provided δ is sufficiently small, depending on n, p. □

7.13 proving rectifiable Reifenberg

Let us finish proving downwards rectifiable Reifenberg. We've shown the inductive hypotheses "manifold covering control", "graphicality", and "hole control" hold at every scale. We've further demonstrated a uniform bound on the compositions:

$$\int_{T_0} |D_{T_0} \tau_i|^p \leq C(n).$$

(of course all provided δ is sufficiently small).

As in the original Reifenberg disc theorem, the maps τ_i are C^0-Cauchy by the refined coarse estimates. Therefore we have a C^0 limit $\tau_\infty : T_0 \to R^n$. By weak-$*$ compactness, and lower semi-continuity of norm, we have that $\tau_\infty \in W^{1,p}(T_0)$ also.

Fix $p > k$. We show most of $\tau_\infty(T_0)$ is contained in a Lipschitz graph. For any a, $\tau_\infty|_{\{|D\tau_\infty| < a\}}$ is Lipschitz, and can be extended to a Lipschitz function $\tilde{\tau}_\infty : T_0 \to R^n$. Write $\tilde{T}_\infty = \tilde{\tau}_\infty(T_0)$.

Given any $A \subset T_0$, we have

$$\mathcal{H}^k(\tau_\infty(A)) = \int_A |J\tau_\infty| \leq c(n) \int_A |D\tau_\infty|^k \leq C(n,p)|A|^{1-k/p}.$$

And therefore

$$\mathcal{H}^k(T_\infty \sim \tilde{T}_\infty) \leq \mathcal{H}^k(\tau_\infty(\{|D\tau_\infty| > a\})) \leq \frac{C(n,p)}{a^{p-k}} \leq \delta,$$

taking $a = a(\delta, n, k)$ sufficiently small.

Take the limit of "manifold covering control" to deduce

$$S \subset T_\infty \cup (\cup_\ell R_\ell \cup E_\ell).$$

By section "bounding μ-mass" we have

$$\mathcal{H}^k(\cup_\ell E_\ell) \leq c\delta,$$

and

$$\begin{aligned}
\mathcal{H}^k(\cup_\ell R_\ell) &\leq \sum_{\ell=1}^{\infty} \sum_{j=1}^{B_\ell} \mathcal{H}^k(S \cap B_{r_\ell}(b_{\ell j})) \\
&\leq 32^k \epsilon_k \sum_\ell B_\ell r_\ell \\
&\leq 320^k \frac{\epsilon_k}{\omega_k}(\mathcal{H}^k T_0 - \mathcal{H}^k T'_\infty + c\delta) \\
&\leq 640^k \epsilon_k + c\delta.
\end{aligned}$$

Therefore, choosing δ sufficiently small, if $\mathcal{H}^k(S \cap B_1) \geq 2^{-k-1}\omega_k$ then

$$\mathcal{H}^k(S \cap B_1 \cap \tilde{T}_\infty) \geq 2^{-k-2}\omega_k \geq \frac{1}{2}\mathcal{H}^k(S \cap B_1).$$

This completes the proof of theorem downwards-rectifiable-Reifenberg.

We now prove rectifiable-Reifenberg. If S is not rectifiable, then there is a purely unrectifiable piece $P \subset S$ of positive (and WLOG finite) $\mathcal{H}^k$-measure. $\mathcal{H}^k \llcorner P$ also satisfies the distortion hypothesis of theorem rectifiable-Reifenberg, since $\mathcal{H}^k \llcorner P \leq \mathcal{H}^k \llcorner S$.

For $\mathcal{H}^k$-a.e. $x \in P$ there is a scale r_x for which $\mathcal{H}^k(P \cap B_{r_x}(x)) \geq 2^{-k-1}\omega_k r_x^k$. Apply downwards-rectifiable-reifenberg to this ball, to deduce that half of $P \cap B_{r_x}(x)$ is rectifiable, i.e. $\mathcal{H}^k(P \cap T_\infty \cap B_{r_x}(x)) \geq \frac{1}{2}\mathcal{H}^k(P \cap B_{r_x}(x))$, for some rectifiable T_∞. This contradicts our choice of P.