1 Varifolds and δV^{tan}

Let S be a C^2 surface, and V an integral n-varifold supported in some open set U. Recall that for any compactly supported C^1 vector field X, we have

$$\delta V(X) = \int div_V(X) d||V||. \tag{1}$$

Let us define δV^{tan} to be δV restricted to the tangential vector fields. Given a vector field X, define the tangential projection

$$X^{TS}(x) = \begin{cases} X(x) & x \notin S \\ \pi_{T_x S} X & x \in S \end{cases}$$

Remark 1.1. In the following, one could drop the integral assumption on V in place of assuming that $||V||(S) = 0$.

Proposition 1.2 (First variation control)**.** *Let $r > 0$ be some radius so that the distance function d to S is smooth in $B_r(S)$. Take $\phi \in C^1_c$ supported in $B_r(S) \cap U$. Suppose δV^{tan} is locally finite, so we can write*

$$\delta V(X) \equiv \delta V^{tan}(X) = \int X \cdot \mu^{tan} d||\delta V^{tan}||, \tag{2}$$

for all tangential vector fields X. Then

$$\lim_{r \to 0} \frac{1}{r} \int_{B_r(S)} \phi |D^T d|^2 d||V|| = -\int_{S^C} D^T \phi \cdot D^T d + \phi div_V(Dd) d||V|| \tag{3}$$

$$+ \int \phi Dd \cdot \mu^{tan} d||\delta V^{tan}|| \tag{4}$$

is a Radon measure on $B_r(S)$.

We deduce that, for any vector field X supported in $B_r(S) \cap U$,

$$\delta V(X) = \int X^{TS} \cdot \mu^{tan} d||\delta V^{tan}|| - \lim_{r \to 0} \frac{1}{r} \int_{B_r(S)} (X \cdot Dd) |D^T d|^2 d||V|| \tag{5}$$

$$- \int_S X \cdot \mathbf{H}_S d||V||. \tag{6}$$

In particular, δV is locally finite and we can bound

$$||\delta V||(B_1) \leq ||\delta V^{tan}||(B_1) + c(n, r_2(S))||V||(B_2). \tag{7}$$

Proof. Take some $\eta : \mathbb{R} \to \mathbb{R}$ C^1 decreasing satisfying $\eta = 0$ on $[1, \infty)$, and $\eta = 1$ on $(-\infty, 1/2]$. Write $\phi(t) = \eta(t/\rho)$, for some $\phi > 0$. Consider the vector field

$$X = h\phi(d) d Dd. \tag{8}$$

Trivially X is tangential. We deduce

$$\delta V^{tan}(X) = \int h\phi(d)dDd \cdot \mu^{tan} d||\delta V^{tan}|| \tag{9}$$

$$= \int D^T h \cdot Dd\phi(d)d + h\phi'(d)d|D^T d|^2 + h\phi(d)|D^T d|^2 + h\phi(d)ddiv_V(Dd)d||V|| \tag{10}$$

$$= \int D^T h \cdot Dd\phi(d)d + h\phi(d)ddiv_V(Dd) + |D^T d|^2 h(\phi - \rho\frac{d}{d\rho}\phi)d||V||. \tag{11}$$

Therefore

$$\frac{d}{d\rho}\left(\frac{1}{\rho}\int h\phi(d)|D^T d|^2\right) = \frac{1}{\rho^2}\int D^T h \cdot Dd\phi(d)d + h\phi(d)ddiv_V(Dd)d|V|| \tag{12}$$

$$-\int h\phi(d)dDd \cdot \mu^{tan} d||\delta V^{tan}||. \tag{13}$$

Integrating this relation between $\tau < \sigma$, and letting $\eta \to 1_{(-\infty,1]}$, we obtain

$$\frac{1}{\sigma}\int_{B_\sigma(S)} h|D^T d|^2 - \frac{1}{\tau}\int_{B_\tau(S)} h|D^T d|^2 \tag{14}$$

$$\int_\tau^\sigma \left(\frac{1}{\rho^2}\int_{B_\rho(S)} D^T h \cdot Ddd + hddiv_V(Dd)d||V|| - hdDd \cdot \mu^{tan} d||\delta V^{tan}||\right) d\rho \tag{15}$$

$$=: \int_\tau^\sigma \frac{1}{\rho^2}\nu\{d < \rho\}d\rho \tag{16}$$

$$= -\frac{1}{\sigma}\nu\{d < \sigma\} + \frac{1}{\tau}\nu\{d < \tau\} + \int_{\sigma < d < \tau}\frac{1}{d}d\nu. \tag{17}$$

Sending $\sigma \to \infty$, and $\tau \to 0$, we obtain

$$\lim_{\tau \to 0} -\frac{1}{\tau}\int_{B_\tau(S)} h|D^T d|^2 = \int_{S^C} D^T h \cdot Dd + hdiv_V(Dd)d||V|| - hDd \cdot \mu^{tan} d||\delta V^{tan}||, \tag{18}$$

which is the required relation (3).

We prove (5). Let $\eta : \mathbb{R} \to \mathbb{R}$ be a non-increasing, non-negative function which is 0 on $[\epsilon, \infty)$. We have

$$\delta V(X) = \delta V(\eta(d)(X\cdot Dd)Dd) \tag{19}$$
$$+\,\delta V^{tan}((1-\eta(d))(X\cdot Dd)Dd + (X-(X\cdot Dd)Dd)) \tag{20}$$
$$= \int div_V(\eta(d)(X\cdot Dd)Dd)d||V|| \tag{21}$$
$$+ \int (X - \eta(d)(X\cdot D)Dd)\cdot \mu^{tan} d||\delta V^{tan}|| \tag{22}$$
$$= \int \eta'(X\cdot Dd)|D^T d|^2 d||V|| + \int \eta div((X\cdot Dd)Dd)d||V|| \tag{23}$$
$$+ \int (X - \eta(d)(X\cdot D)Dd)\cdot \mu^{tan} d||\delta V^{tan}|| \tag{24}$$
$$=: I_1 + I_2 + I_3. \tag{25}$$

Let η tend to the function $t \mapsto 1 - \max(0, t/\epsilon)$, and then take $\epsilon \to 0$. We find that

$$I_1 = \lim_{\epsilon\to 0} -\frac{1}{\epsilon}\int_{B_\epsilon(S)} (X\cdot Dd)|D^T d|^2. \tag{26}$$

Write ν_S for a choice of normal for S. Since V is integral, $T_x V = T_x S$ for $||V||$-a.e. $x \in S$, and therefore

$$I_2 = \int_S div_S((X\cdot \nu_S)\nu_S)d||V|| = -\int_S H_S \cdot X d||V||, \tag{27}$$

and

$$I_3 = \int X^{TS}\cdot \mu^{tan} d||\delta V^{tan}||. \tag{28}$$

This proves (5). To deduce the final equation (7), we use that (3) is a Radon measure, and that we can take $r = \epsilon(n) r_2(S)$. □

Proposition 1.3 (First variation structure)**.** *If V and δV are locally finite, and we write*

$$\delta V(X) = \int X\cdot \mu d||\delta V||, \tag{29}$$

for $|\mu| = 1$, then

$$\delta V^{tan}(X) = \int X \cdot \mu^{TS} d||\delta V||, \tag{30}$$

and $||\delta V^{tan}|| = |\mu^{TS}|||\delta V||$. In particular, there is a norm-preserving extension of δV^{tan} to the space of all compactly supported C^1 vector fields.

Remark 1.4. We shall henceforth identify δV^{tan}, defined in (1) as an operator only on tangential fields, with its norm-preserving extension (30) to all vector fields.

Proof. Since

$$\delta V(Y) = \int Y \cdot \mu^{TS} d||\delta V|| \tag{31}$$

for any tangential Y, we trivially have $||\delta V^{tan}|| \leq |\mu^{TS}|||\delta V||$. We show the converse. Fix an open set U', and WLOG assume $\partial U'$ is smooth. Choose a sequece $X_i \in C^1_c(U', \mathbb{R}^{n+k})$, $|X_i| \leq 1$, so that

$$\delta V(X_i) \to ||\delta V||(U'). \tag{32}$$

Since $|\mu| = 1$ also, we must have $X_i \to \mu$ $||\delta V||$-a.e. in U', and hence $X_i^{TS} \to \mu^{TS}$ $||\delta V||$-a.e. in U' also.

Let $\eta(t) : \mathbb{R} \to \mathbb{R}$ be a smooth function which is 0 for $t \geq 1$, and 1 for $t \leq 1/2$. Pick a sequence $\delta_i \to 0$, and define $\eta_i(t) = \eta(t/\delta_i)$. If $Z_i : S \to \mathbb{R}^{n+k}$ is the vector field

$$Z_i = (1 - \eta_i(|X_i^{TS}|)) \frac{X_i^{TS}}{|X_i^{TS}|}, \tag{33}$$

then $Z_i \in C^1(S, \mathbb{R}^{n+k})$, spt$Z_i \subset S \cap U'$, $|Z_i| \leq 1$, and $Z_i \to \mu^{tan}/|\mu^{tan}|$ $||\delta V||$-a.e. $x \in S \cap U'$. In particular, by the DCT, we have

$$\int_S Z_i \cdot \mu d||\delta V|| \to \int_{S \cap U'} |\mu^{TS}| d||\delta V||. \tag{34}$$

Let d_S be the distance function to S, and $d_{U'}$ the positive (in U') distance function to $\partial U'$. Let $\xi(x)$ be the nearest-point projection to $\partial\Omega$, which is well-defined and smooth in some neighborhood of $\partial\Omega$. For large i, we can define the fields

$$Y_i(x) = \eta_i(d_{U'}(x))(\eta_i(d_S(x))Z_i(\xi(x)) + (1 - \eta_i(d_S(x))X_i(x)). \tag{35}$$

Then (for i large), $Y_i \in C^1$, spt$Y_i \subset U'$, $|Y_i| \leq 1$, and Y_i is tangential. Moreover, $Y_i \to \mu^{TS}/|\mu^{TS}|$ wherever $X_i \to \mu$ and $\mu \neq 0$, and hence by the DCT we get

$$\int Y_i \cdot \mu d||\delta V|| \to \int_U |\mu^{TS}| d||\delta V||. \tag{36}$$

But Y_i is tangential, and so the LHS is $\delta V(Y_i) = \delta V^{tan}(Y_i)$. This proves $||\delta V^{tan}|| \geq |\mu^{TS}|||\delta V||$, and complete the proof of the Lemma. □

Corollary 1.5. *If V and δV^{tan} are locally finite and $||\delta V^{tan}|| << ||V||$, then we can write*

$$\delta V(X) = -\int X \cdot H d||V|| + \int X \cdot \nu_S d\sigma, \tag{37}$$

for some measure $\sigma \perp ||V||$, supported in S. Similarly, we can write

$$\delta V^{tan}(X) = -\int X \cdot H^{TS} d||V|| =: -\int X \cdot H^{tan} d||V||. \tag{38}$$

Proof. Follows directly from Proposition 1.3, since $\nu_S^{TS} \equiv 0$. □

Corollary 1.6. *If $S = \partial\Omega$, V and δV^{tan} are locally finite, $||\delta V^{tan}|| << ||V||$, and $\mathrm{spt} V \subset \overline{\Omega}$, then we can write*

$$\delta V(X) = -\int X \cdot H d||V|| + \int X \cdot \nu_\Omega d\sigma, \tag{39}$$

where ν_Ω is the outer unit normal to Ω, and $\sigma \perp ||V||$ is a non-negative measure supported in $\partial\Omega$. In particular, for any $\phi \in C^1_c(U)$, we have

$$\int \phi d||\delta V|| \leq c(n, r_2(S)) \int |\phi| + |D^T \phi| d||V||. \tag{40}$$

Proof. From Corollary REF, we can write

$$\delta V(X) = -\int X \cdot H d||V|| + \int X \cdot \eta d\sigma,$$

where $\sigma \perp ||V||$ is a non-negative measure supported in S, and $\eta(x) = \pm\nu_\Omega(x)$. We must show that $\eta = \nu_\Omega$.

Let $\tilde{d}$ be the signed distance function to $\partial\Omega$, which is > 0 in Ω. □

Proposition 1.7 (Compactness)**.** *Let V_i be a sequence of integral varifolds, S_i a sequence of C^2 surfaces, so that $S_i \to S$ in C^2 and*

$$\sup_i ||V_i||(U') + ||\delta V_i^{tan_i}||(U') < \infty \quad \forall U' \subset\subset U. \tag{41}$$

Here $\delta V_i^{tan_i}$ is the tangential variation w.r.t. to the surface S_i.

Then after passing to a subsequence, there is an integral varifold V in U, so that $V_i \to V$. Moreover, $\delta V_i^{tan_i} \to \delta V^{tan}$, in the sense that the extensions of (30) *converge:*

$$\int X \cdot \mu_i^{tan_i} d||\delta V_i^{tan_i}|| \to \int X \cdot \mu^{tan} d||\delta V^{tan}|| \tag{42}$$

for all vector fields $X \in C^1_c$. As a consequence,

$$||\delta V_i^{tan}|| \leq \liminf_i ||\delta V_i^{tan_i}||.$$

Proof. From Proposition 1.2, we have uniform local control over $||\delta V_i||$, and so the first statement follows by Allard's compactness theorem. We prove (42). Since $\delta V = \delta V^{tan}$ away from S, it suffices to consider vector fields supported in some neighborhood $B_r(S) \cap U$. Write d, d_i for the distance functions to S, S_i, and WLOG we can choose r so that d, d_i are smooth on $B_r(S) \cap U$.

Take X C^1_c, supported in $B_r(S) \cap U$. From Proposition 1.2, we can write

$$\int X \cdot \mu^{tan} d||\delta V^{tan}|| = \int (X - (X \cdot Dd)Dd) \cdot \mu d||\delta V|| \tag{43}$$

$$+ \int_{S^C} D^T(X \cdot Dd) \cdot Dd + (X \cdot Dd) div_V(Dd) d||V|| \tag{44}$$

$$- \int_S X \cdot H_S d||V|| \tag{45}$$

$$=: I(X, V, S), \tag{46}$$

and similarly with V, S, d replaced with V_i, S_i, d_i.

Since $S_i \to S$ in C^2, we have that $Dd_i \to Dd$ in C^1_{loc}, and $H_{S_i} 1_{S_i} \to H_S 1_S$ pointwise, and is uniformly bounded. Therefore, by the DCT we deduce

$$|I(X, V_i, S_i) - I(X, V_i, S)| \to 0.$$

It then follows by varifold convergence $V_i \to V$, $\delta V_i \to \delta V$, that $I(X, V_i, S) \to I(X, V, S)$. This proves the weak convergence (42), and lower-semi-continuity of norm follows directly. □

Corollary 1.8 (L^p compactness)**.** *Let V_i be a sequence of integral n-varifolds, S_i a sequence of C^2 surfaces, and $1 < p \leq \infty$. Suppose that $S_i \to S$ in C^2, and $||\delta V_i^{tan_i}|| << ||V_i||$, and*

$$\sup_i ||V_i||(U') + \left(\int_{U'} |H_i^{tan_i}|^p d||V_i|| \right)^{1/p} < \infty \quad \forall U' \subset\subset U.$$

Then there is an integral n-varifold V, with $||\delta V^{tan}|| << ||V||$, so that $V_i \to V$, and

$$\int X \cdot H_i^{tan_i} d||V_i|| \to \int X \cdot H^{tan} d||V|| \quad \forall X \in C^1_c,$$

and

$$\left(\int_{U'} |H^{tan}|^p d||V|| \right)^{1/p} \leq \liminf_i \left(\int_{U'} |H_i^{tan_i}|^p d||V_i|| \right)^{1/p} \quad \forall U' \subset\subset U.$$

Proof. By Corollary 1.5, the main thing we need to check is that $||\delta V^{tan}|| << ||V||$. We observe that, for every X supported in $U' \subset\subset U$,

$$|\delta V_i^{tan_i}(X)| = \left| \int H_i^{tan_i} \cdot X d||V_i|| \right| \tag{47}$$

$$\leq \left(\int_{U'} |H_i^{tan_i}|^p d||V_i|| \right)^{1/p} \left(\int |X|^{p/(p-1)} d||V_i|| \right)^{(p-1)/p}. \tag{48}$$

Taking the limit on both sides, we deduce that

$$|\delta V^{tan}(X)| \leq \liminf_i \left(\int_{U'} |H_i^{tan_i}|^p d||V_i|| \right)^{1/p} \left(\int_{U'} |X|^{p/(p-1)} d||V|| \right)^{(p-1)/p}.$$

Therefore δV^{tan} is a bounded linear operator on $L^{p/(p-1)}(U'; ||V||)$. By the Riesz representation theorem it follows that

$$\delta V^{tan}(X) = -\int X \cdot H^{tan} d||V||,$$

with $||H^{tan}||_{L^p(U';||V||)} \leq \liminf_i ||H_i^{tan_i}||_{L^p(U';||V_i||)}$. $\square$