Notes from Brian White's class on minimal surfaces

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Plateau problem: Given Γ a simple closed curve in R^N, what is a (the?) surface of least area with $\partial M = \Gamma$?

Why is existence hard? The natural method is the direct method: let

$$\alpha = \inf\{|M| : M \text{ is a disc with } \partial M = \Gamma\}.$$

Take a minimizing sequence M_i. The hope is that one can find a convergent subsequence, BUT there also exist some very bad minimizing sequences. E.g. even if Γ is a circle, we can have "spacefilling" minz sequences like PICTURE

1 Douglas-Rado solution

Write D^2 for the open disc in R^2. Let

$$\mathcal{C}_\Gamma = \left\{ f : \bar{D}^2 \to R^N : \begin{array}{l} f \text{ continuous on } \bar{D}, \text{ smooth or loc. Lipschitz on } D \\ \text{such that } f(\partial D) = \Gamma \text{ monotonically} \end{array} \right\}$$

The area of $f(D)$ is defined to be

$$A(f) = \int_D |JDf| = \int_D \sqrt{|f_x|^2|f_y|^2 - (f_x \cdot f_y)^2}\, dxdy.$$

Notice that A is independent of parameterization, i.e. $A(f) = A(f \circ u)$ for any diffeomorphism $u : D \to D$.

Theorem 1.1. *There exists an $F \in \mathcal{C}_\Gamma$ such that:*

- $A(F) = \alpha = \inf\{A(G) : G \in \mathcal{C}_\Gamma\}$
- *F is harmonic (i.e. $\Delta F = 0$)*
- *F is almost conformal, in that $|F_x| = |F_y|$ and $F_x \cdot F_y = 0$*
- *$\{p : |DF(p)| = 0\}$ is isolated .*

1.2 energy

Given $f : D \to R^N$, the energy of f is defined to be

$$E(f) = \frac{1}{2}\int |Df|^2 = \frac{1}{2}\int |f_x|^2 + |f_y|^2.$$

Proposition 1.3. *$A(F) \leq E(F)$ with equality iff F is almost conformal.*

Proposition 1.4. *Suppose $f, h : \bar{D} \to R^N$ agree on ∂D, with h harmonic. Then $E(f) \geq E(h)$.*

Proof. Write $f - h = \phi$, so $\phi = 0$ on ∂D. We have

$$\begin{aligned} E(f) &= \frac{1}{2}\int |Dh + D\phi|^2 \\ &= E(h) + \int Dh \cdot D\phi + E(\phi) \\ &= E(h) + \int_{\partial D} \partial_n h\phi - \int_D \Delta h\phi + E(\phi) \\ &\geq E(h). \end{aligned}$$

□

Proposition 1.5. *If $F : D \to R^N$ and $u : D \to D$ is a diffeomorphism, then $A(F) = A(F \circ u)$, and $E(F) = E(F \circ u)$ if u is conformal.*

Proof of the Theorem. Step 1: inf is the same if we strict to only smooth maps.

Idea: given locally Lipschitz $f \in \mathcal{C}_\Gamma$, we can approx by a smooth map $\tilde{f}$ with $A(\tilde{f}) \leq A(f) + \epsilon$. (just mollify)

Step 2: given smooth $f \in \mathcal{C}_\Gamma$ and $\epsilon > 0$, $\exists$ harmonic $h \in \mathcal{C}_\Gamma$ such that $E(h) \leq A(f) + \epsilon$.

Case 1: Suppose f maps $\bar{D}$ diffeomorphically to $f(\bar{D})$. Then by Morrey (REF) there is smooth conformal diffeomorphism $g : \bar{D} \to f(\bar{D})$. Thus

□

2 monotonicity

Definition 2.0.1. M is minimal $\iff \frac{d}{dt}|_{t=0}|\phi_t(M)| = 0$ for all ϕ_t generated by X compactly supported in $M\tilde{\partial}M \iff H = 0$.

Corollary 2.1. *If M is a compact, minimal m-surface in R^N, then*

$$m|M| = \int_{\partial M} X \cdot \eta.$$

Let $C = \text{cone}_0(\partial M)$. Since $H_C \perp X$, we have similarly

$$m|C| = \int_{\partial C} X \cdot \eta_C.$$

But $X \cdot \eta_C \geq X \cdot \eta$ along $\partial M = \partial C$, and so

$$|M| \leq |\text{cone}_0(\partial M)|.$$

Now take $A(r) = |M \cap B_r(p)|$, for $B_r(p) \cap \partial M = \emptyset$. By the coarea formula we have

$$A'(r) \geq |\partial(M \cap B_r(p))| = |M \cap \partial B_r(p)|.$$

And by the previous calculation, we have

$$A(r) \leq |\text{cone}_p(M \cap \partial B_r(p))| = \frac{r}{m}|M \cap \partial B_r(p)|.$$

Therefore we deduce that $A'(r) \geq \frac{m}{r}A(r)$. Integrating this, we obtain

Theorem 2.2 (monotonicity)**.** *We have*

$$\frac{|M \cap B_r(p)|}{r^m} \text{ is increasing in } r$$

for $0 < r < \text{dist}(p, \partial M)$. *This quantity is strictly increasing inles* M *is a cone over* p.

The above quantity is the density ratio of M, and its monotonicity is the most important tool in minimal surface theory.

Definition 2.2.1. Set

$$\Theta(M, p, r) = \frac{|M \cap B_r(p)|}{\omega_m r^m}.$$

Now let E_p be the exterior cone at p over ∂M. I.e.

$$E_p = \{\lambda x : x \in \partial M, \lambda \geq 1\}.$$

By applying the first-variation for $X(x) = x$ to both M and E separately, and adding the result, we obtain

Theorem 2.3 (extended monotonicity)**.** *We have that*

$$\Theta(M \cup E_p, p, r)$$

is increasing for all r.

Theorem 2.4. *Let* M *be a minimal submanifold of* R^N *with* ∂M *compact (or empty). Then the "density at infinity" of* M

$$\Theta(M) := \lim_{r \to \infty} \Theta(M, p, r)$$

exists, is independent of p, *and is* ≥ 1 *if* M *is unbounded.*

Proof. We can assume ∂M is smooth, otherwise replace M by $M \sim B_r(p)$ for some appropriate r. Let E_p be the exterior cone at p over ∂M, so that $\Theta(M \cup E_p, p, r)$ is an increasing function of r, and in particular

$$\lim_{r\to\infty} \Theta(M \cup E, p, r) = \lim_{r\to\infty} (\Theta(M, p, r) + \Theta(E, p, r))$$

exists.

Let C_p be the entire cone at p over ∂M. Then for all r,

$$\Theta(C_p, p, r) = const = \Theta(E_p, p, r) + \Theta(C \sim E, p, r).$$

Since $C \sim E$ is compact, we know $\lim_{r\to\infty} \Theta(E, p, r) = \Theta(C_p, p, 1) < \infty$. Hence $\Theta(M, p, r)$ has a (possibly infinite) limit.

We show the limit is independent of p. For any other q, we have

$$\frac{|M \cap B_r(p)|}{r^m} \leq \frac{|M \cap B_{r+|p-q|}(q)|}{(r + |p - q|)^m} \frac{r^m}{(r + |p - q|)^m}.$$

Take $r \to \infty$ to deduce that $\Theta(M, p, \infty) \leq \Theta(M, q, \infty)$. But p and q are arbitrary, and so the opposite inequality also holds.

We show $\Theta(M) \geq 1$ when M is unbounded. If $\partial M = \emptyset$, then We have that

$$\Theta(M, p) := \lim_{r\to 0} \Theta(M, p, r)$$

exists and is clearly equal to the number of sheets of M passing through M (simply because M is smooth). So taking $p \in M$ gives $\Theta(M) \geq \Theta(M, p) \geq 1$.

If $\partial M \neq \emptyset$, take $q \in M$ at distance $\geq R$ away from ∂M. Taking E_q the exterior cone of ∂M over q, we have

$$1 \leq \Theta(M, q) \leq \Theta(M \cup E_q) = \Theta(M) + \Theta(E_q).$$

But $\Theta(E_q) = \Theta(C_q) \to 0$ as $R \to \infty$. □

Corollary 2.5. *If $M \sim B_R(0)$ consists of Q unbounded components, then $\Theta(M) \geq Q$.*

Proof. For each component E, we apply the above theorem to deduce $\Theta(E) \geq 1$. Then we have

$$\Theta(M) \geq \sum_{\text{components } E} \Theta(E) \geq Q.$$

□

Definition 2.5.1. M has an end E if for every large radius R, $E \sim B_R$ is a connected (unbounded) component of $M \sim B_R$.

The previous corollary implies that if M is minimal and has $\Theta(M) < \infty$, then necessarily M has a finite number of ends.

3 limits of minimal surfaces

Suppose we have a sequence of minimal surfaces $M_i \subset \Omega$ with $\partial M_i \subset \partial\Omega$. When do we have subsequential limits, and what can we say about them?

We remark that all the below theorems are valid in a Riemannian manifold, being local in nature.

Theorem 3.1 (theorem 0). *If each $M_i = \text{graph} f_i$ with $f_i : B^m \to R^{N-m}$, with*

$$\sup_i |f_i|_{C^2} < \infty,$$

then (subseq) $f_i \to f$ smoothly on compact subsets on B^m, where $M = \text{graph} f$ is a smooth minimal surface.

Proof. The hypothesis gives subsequential convergence in $C^{1,\alpha}$ to f, and Schauder theory boosts regularity up to C^∞. □

We introduce some notation. Given $p \in M_i$, let $\beta(M_i, p)$ be the norm of the second fundamental form of M_i at p.

Theorem 3.2 (theorem 1). *Take $M_i \subset \Omega$ minimal, with $\partial M_i \subset \partial\Omega$. Suppose we have uniform area and curvature bounds on compact sets. I.e. for all $K \subset\subset \Omega$ we have*

$$|M_i \cap K| \leq A_K < \infty, \quad \sup_K \beta(M_i, \cdot) \leq C_K < \infty,$$

where of course A_K and C_K are independent of i. Then (subseq) we have C^∞ convergence $M_i \to M$ to a minimal surface M.

Proof. By uniform bounds on curvature, in a small enough ball around any point all the $\{M_i\}$ are (a finite union of) graphs with uniform C^2 bounds. The uniform area bounds imply the number of graphs required for each M_i is uniformly bounded. Now in each of these balls we can apply theorem 0. □

Theorem 3.3 (easy theorem). *Take $M_i \subset \Omega$ minimal, with $\partial M_i \subset \partial\Omega$, and locally uniformly bounded area. Suppose $\beta(M_i, \cdot)$ has a local maximum at p_i, with $p_i \to p \in \Omega$ and $\beta(M_i, p_i) \to \infty$. Then (subseq)*

$$\lambda_i(M_i - p_i) \to M \text{ smoothly,}$$

where $\lambda_i = \beta(M_i, p_i)$. Here M is a minimal surface in R^N having $\partial M = \emptyset$, and with

$$\max \beta(M, \cdot) = \beta(M, 0) = 1.$$

Proof. On any compact set K, provided $i \geq i_0(K)$, each $M_i' = \lambda_i(M_i - p_i)$ has maximum curvature $= 1$ on K (attained at 0), and trivially has uniformly bounded area. Now apply theorem 1, and observe that

$$\text{dist}(0, \partial M) = \lim_i \lambda_i \text{dist}(p_i, \partial M_i) \geq \lim_i \lambda_i \text{dist}(p_i, \partial\Omega) \to \infty.$$

□

By choosing our blow-up sequence a little more carefully we can avoid the hypothesis that p_i locally maximizes β.

Theorem 3.4. *Take $M_i \subset \Omega$ min, with $\partial M_i \subset \partial\Omega$, and uniform area bounds near p. Let $p_i \in M_i$ be such that $p_i \to p$ and $\beta(M_i, p_i) \to \infty$.*

Then (subseq) $\exists q_i \in M_i$ such that $q_i \to p$, $\beta(M_i, q_i) \to \infty$, and

$$\beta(M_i, q_i)(M_i - q_i) \to M \text{ smoothly},$$

where M is a smooth minimal surface in R^N having $\partial M = 0$, and $\max \beta(M, \cdot) = \beta(M, 0) = 1$.

Proof. Choose $r_i \to 0$ so that $B_{r_i}(p_i) \subset \Omega$, and $r_i\beta(M_i, p_i) \to \infty$, and $\Theta(M_i, p, r_i) \leq 2$. Let q_i maximize

$$\beta(M_i, \cdot)\mathrm{dist}(\cdot, \partial B_{r_i}(p_i))$$

in $B_{r_i}(p_i)$. Let $R_i = \mathrm{dist}(q_i, \partial B_{r_i}(p_i))$. Then q_i also maximizes

$$\beta(M_i, \cdot)\mathrm{dist}(\cdot, \partial B_{R_i}(q_i)),$$

simply because $\mathrm{dist}(\cdot, \partial B_{R_i}(q_i)) \leq \mathrm{dist}(\cdot, \partial B_{r_i}(p_i))$.

Now $q_i \to p$ since $r_i \to 0$, and $R_i\beta(M_i, q_i) \geq r_i\beta(M_i, p_i) \to \infty$. Let

$$M_i' = \lambda_i(M_i - q_i),$$

where $\lambda_i = \beta(M_i, q_i)$.

By construction $\beta(M_i', 0) = 1$, and since the quantities we maximized with are scale-invariant, we have

$$\beta(M_i', x) \leq \frac{\lambda_i R_i}{\lambda_i R_i - |x|} \to 1.$$

So M_i' has uniform local curvature bounds.

Uniform local area bounds follow by upper-semicontinuity of density.

$$\begin{aligned}
\limsup_i |M_i' \cap B_R(0)| &= \limsup_i \lambda_i^n |M_i \cap B_{R/\lambda_i}(q_i)| \\
&= \limsup_i R^n \Theta(M_i, q_i, R/\lambda_i) \\
&\leq \Theta(\mu, x)
\end{aligned}$$

(in a manifold we would have an exponential error term, depending on the local geometry of Ω near p)

Now apply theorem 1 to obtain a subsequential limit $M_i' \to M$. We know M lives in R^N since $\lambda_i \to \infty$. The fact that $\partial M = \emptyset$ follows from

$$\mathrm{dist}(0, \partial M_i') \geq \lambda_i R_i \to \infty.$$

□

Let's continue to work with M_i as in the previous theorem, *but* assume further that M_i have uniform local area bounds. Each M_i gives rise to a Radon measure via $|M_i|(U) = |M_i \cap U|$. Of course $|M_i|$ is the varifold mass measure induced by M_i interpreted as a varifold. Since $|M_i|$ is uniformly bounded on compact sets, then (subseq) $|M_i| \rightharpoonup \mu$ as measures, for μ a Radon measure.

Suppose $M_i \ni x_i \to x$. Since $\Theta(M_i, x_i, r)$ is increasing in r, then $\Theta(\mu, x, r)$ is increasing in r also. In particular we have a well-defined limit

$$\Theta(\mu, x) := \lim_{r \to 0} \Theta(\mu, x, r) < \infty.$$

Lemma 3.5. *Θ is upper-semicontinuous, in the sense that if $|M_i| \rightharpoonup \mu$, $x_i \to x$ and $r_i \to 0$, then*

$$\limsup_i \Theta(M_i, x_i, r_i) \leq \Theta(\mu, x).$$

Proof. Fix an r, and let $\delta_i = |x - x_i|$. We know that (for a.e. r)

$$|M_i|(B_r(x)) \to \mu B_r(x),$$

and for i sufficiently large, we have

$$|M_i|(B_r(x)) \geq |M_i|(B_{r-\delta_i}(x_i)) \geq (r - \delta_i)^n \Theta(M_i, x_i, r_i).$$

Therefore, for a.e. r, we have

$$\limsup_i \Theta(M_i, x_i, r_i) \leq \Theta(\mu, x, r).$$

Now take $r \to 0$. □

Let M' be as in the previous theorem, i.e. so we had a $\lambda_i \to \infty$ and an $x_i \to x$ such that $\lambda_i(M_i - x_i) \to M'$.

What the following theorem says is tantamount to: the (smooth) blow-up limit of μ at x has density-at-infinity $\leq \Theta(\mu, x)$, where μ is realized as the Radon limit of smooth M_i. Secretly we are saying that the blow-up of a stationary integral varifold at a point has density $\leq$ density at that point, but stating everything as limits of smooth guys greatly reduces the necessary framework.

Theorem 3.6. *We have $\Theta(M') \leq \Theta(\mu, x)$.*

Proof. For a.e. R, we have

$$\begin{aligned} \Theta(M', 0, R) &= \lim_i \Theta(M'_i, 0, R) \\ &= \lim_i \Theta(M_i, x_i, R/\lambda_i) \\ &\leq \Theta(\mu, x). \end{aligned}$$

□

In particular, we have that M' has Euclidean area growth.

Corollary 3.7. *$\Theta(M') < \infty$.*

Theorem 3.8 (easy Allard)**.** *If the curvatures of M_i blow up at x, then $\Theta(\mu, x) > 1$.*

Proof. Of course we can assume $\Theta(\mu, x) < \infty$, implying we have local area bounds near x. So apply theorem blow-up-limit, and we know that $\lambda_i(M_i - x_i) \to M'$ smoothly, and that $\Theta(M') \leq \Theta(\mu, x)$. Trivially $\Theta(M') \geq \Theta(M', 0) \geq 1$, with equality iff M' is a plane (cone at origin + smooth). But by construction the curvature of M' at 0 is 1, so we must have strict inequality $\Theta(M') > 1$. □

Theorem 3.9. *If M' is a complete minimal surface in R^N, with $\Theta(M') < \infty$, and*

$$\max |B_{M'}(\cdot)| = |B_{M'}(0)| = 1,$$

then $\Theta(M') \geq 1 + \epsilon(m, N)$.

Proof. Let

$$\alpha = \inf\{\Theta(M') : M' \text{ as in hypothesis}\}.$$

Choose a sequence M'_i such that $\Theta(M'_i) \to \alpha$. We have curvature bounds and local area bounds by assumption, so we can take a subsequential limit $M'_i \to M'$ smoothly. M' will be a miniaml surface satisfying the hypotheses.

Now for a.e. R, we have

$$\Theta(M', 0, R) = \lim_i \Theta(M'_i, 0, R) \leq \lim_i \Theta(M'_i) = \alpha,$$

and so $\Theta(M') \leq \alpha$. But M' is non-flat and smooth at the origin, and so $\Theta(M') > 1$. □

Remark 3.10. We note that we can have strict inequality in theorem limit-surface-bound. We give an example. Let M_g be the Coast-Hoffman-Meeks surface of genus g. This is a minimal surface asymptotic to the union of a catenoid and a plane. Set

$$M'_i = \frac{1}{g} M_g.$$

The curvature will blow up at the handles, so the smooth limit surface M' will be Sherks surface, which is a desingularization of two planes. In particular, $\Theta(M') = 2$.

However μ is the Radon measure associated to a multiplicity-3 plane, and so $\Theta(\mu, x) = 3$.

Here's an exercise. Suppose $\dim M' = 2$, and $\Theta(M') < \infty$. Show that $\Theta(M')$ is an integer. Or, as a special case, show that

$$\alpha = \inf_{M' \text{ not flat}} \Theta(M') = 2.$$

Or, suppose that $|M_i| \rightharpoonup \mu$, and $\mu(U) = \theta |U \cap P|$ for some number θ and some plane P. Then show θ must be an integer.

Remark 3.11. Notice that if $M \subset M'$, then genus$(M) \leq$ genus(M'). Therefore, if $\dim M_i = 2$, with genus$(M_i) \leq g$, then M', as a smooth limit of blow-ups, will have genus$(M') \leq g$ also.

4 genus and total curvature

Lemma 4.1. *Suppose $M^2 \subset R^N$ is a compact minimal surface. Then*

$$\max_M \operatorname{dist}(\cdot, \partial M) \leq \frac{\text{length}\partial M}{2\pi}.$$

Proof. Take $p \in M \sim \partial M$, and let E be the exterior cone of ∂M over p. We have

$$1 \leq \Theta(M, p) \leq \Theta(M \cup E, p, \infty) = \Theta(C, p, 1).$$

But

$$\begin{aligned} \Theta(C, p, 1) &= \frac{1}{\pi} \int_0^1 |C \cap \partial B_r(0)| dr \\ &= \frac{|\text{proj}_{\partial B_1} \partial M|}{2\pi} \\ &\leq \frac{|\partial M|}{2\pi}. \end{aligned}$$

□

Corollary 4.2. *Suppose M minimal is connected and bounded by two curves Γ_1, Γ_2. Then* $\operatorname{dist}(\Gamma_1, \Gamma_2) \leq \frac{|\partial M|}{2\pi}$.

Proof. Choose a point p lying in $M \cap \{x : \operatorname{dist}(x, \Gamma_1) = \operatorname{dist}(x, \Gamma_2)\}$, then apply the above theorem. □

Theorem 4.3. *Suppose M^2 is minimal in R^N, properly immersed, with finite genus and quadratic area growth. Then M has finite total curvature, i.e.*

$$\int_M |K| < \infty.$$

Remark 4.4. If you obtain M as a blow-up limit, so that you know M has quadratic area growth and pointwise bounded curvature (e.g. as in theorem blow-up-limits), then M must be properly immersed.

Remark 4.5. Recall that (for a 2-dim surface) $|A|^2 = H^2 - 2K$. In particular, if M^2 is minimal, then

$$K = -|A|^2/2.$$

Proof. Case 1: M is simply-connected.

Take $p \in M$, and let

$$M(r) = \{x \in M : \operatorname{dist}_M(x, p) \leq r\} \subset B_r(p).$$

Since M is intrinsically a Hadamard space, $\partial M(r)$ is smooth for all r. Write $A(r) = |M(r)|$ and $L(r) = |\partial M(r)|$. We have

$$A'(r) = L(r)$$

and

$$\begin{aligned}
A''(r) &= L'(r) \\
&= \int_{\partial M(r)} kds \quad (k \text{ being the curvature of } \partial M \subset M) \\
&= 2\pi - \int_{M(r)} KdA \\
&= 2\pi + \int_{M(r)} |K|dA.
\end{aligned}$$

Now if $\int_{M(r)} |K|dA \to \infty$ as $r \to \infty$, we would have $A''(r) \to \infty$, and hence

$$\Theta(M, 0, r) \geq \frac{A(r)}{\pi r^2} \to \infty.$$

This is a contradiction.

Case 2: general M.

As remarked at the end of section monotonicity, quadratic area growth ($\iff$ $\Theta(M) < \infty$) implies M has finitely many ends. Finite genus implies each end is annular. So M is homemorphic (in fact conformal) to a compact surface minus finitely many points.

Take E a minimal annulus properly immersed in R^N. Choose Γ some curve in E, and take a sequence of curves $\Gamma_i \to \infty$ (in E) homotopic to Γ. By the previous lemma, we have

$$2\pi \text{dist}(\Gamma, \Gamma_i) \leq |\Gamma| + |\Gamma_i|,$$

and hence (by properness) $|\Gamma_i| \to \infty$. We are therefore justified in replacing Γ with the shortest curve (geodeisc) in M homotopic to Γ. We can assume $\partial E = \Gamma$.

Let $E(r) = \{x \in E : \text{dist}(x, \Gamma) \leq r\}$, and write $A(r) = |E(r)|$, $L(r) = |\partial E(r)|$. Since the universal cover of E is a half a Hadamard space, bounded by the lift of Γ, we know $\partial E(r)$ is smooth for all r.

Recalling that Γ is a geodesic, we calculate

$$A''(r) = L'(r) = \int_{\partial E(r)} kds = \int_{E(r)} |K|dA.$$

As before this must be finite, otherwise we contradict extrinsic quadratic area growth. □

Theorem 4.6. *Take $M^2 \subset R^N$ complete, minimal, finite total curvature. Then:*

A) M is conformally a punctured compact surface $\Sigma \sim \{p_1, \ldots, p_k\}$.

B) $\text{Tan}(M, p)$ *extends continuously (meromorphically, if $N = 3$) to Σ.*

C)

$$-\int KdA = \int |K|dA = \begin{cases} 4\pi j & (R^3) \\ 2\pi j & (R^N) \end{cases}.$$

D) M proper, with quadratic area growth.

Proof. A) Is in fact true for any complete M with $\int |K^-| < \infty$. (see Brian's paper)

B) We prove this for R^3. Look at the normal map $\nu : M \to S^2$. Then $|K|$ is precisely the Jacobian for ν, so that for $U \subset M$ we have

$$|\nu(U)| = \int_U |K| dA.$$

We recall that M is minimal precisely when ν is conformal and orientation reversing. To see this, pick an ON basis which diagonalizes $D\nu$ as an automorphism of T_pM, and we have $H_p = 0$ iff

$$D\nu|_p = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

Let $p \in \{p_1, \ldots, p_k\}$. Choose a punctured disc $E \ni p$ so that $|\nu(E)| = \int_E |K| < 2\pi < |S^2|$. Since $\nu : E \to S^2$ must miss at least three points, the composition

$$g = (\text{Riemann projection}) \circ \nu : E \to C$$

is holomorphic and misses two points. Therefore g extends meromorphically to p.

C) Follows by degree theory. Since ν extends as a continuous map $\Sigma \to S^2$, we have

$$\int_\Sigma |K| = -\int_\Sigma \det(D\nu) = -\deg(\nu)|S^2|.$$

(recall ν reverses orientation)

D) We prove properness. Choose an annulur end E about p_j, with ∂E compact. We can suppose $\lim_{p \to p_j} \text{Tan}(E, p)$ is horizontal, and hence assume $\text{Tan}(E, \cdot)$ has slope ≤ 1. Parameterize E by horizontal projection to a horizontal plane π. Then we have

$$\frac{1}{\sqrt{2}} \text{dist}_M(p, \partial E) \leq \text{dist}_\pi(\text{proj}_\pi(p), \text{proj}_\pi(\partial E)) \leq \text{dist}_{R^N}(p, \partial E).$$

So if $q_i \to \infty$ in M, we must have $q_i \to \infty$ in R^N also.

We prove quadratic area growth. Take E as before, then by properness and smallness assumption on slope we know that for some R,

$$\text{proj}_\pi : E \sim (B_R^2 \times R^{N-2}) \to R^2 \sim B_R,$$

is a ℓ-sheeted covering map. □

Remark 4.7. Using the notation in the above proof, if $M^2 \subset R^3$ and $E \sim (B_R^2 \times R)$ is embedded, then actually $E \sim (B_R^2 \times R)$ is a graph. To see this, simply follow a large curve in E around the origin. If ever E is two-valued, then E must stay two-valued by connectedness.

However, this is false if $N > 3$. For example, in $R^4 = C^2$, the surface defined by $\{(z, w) : z = w^2\}$ has an embedded, two-valued end.

More succinctly, we have proven the following.

Theorem 4.8. *Take $M^2 \subset R^N$ minimal, immersed. The following are equivalent:*

1) M proper, quadratic area growth, finite genus.

2) M complete, finite total curvature.

Further, for M satisfying either 1) or 2), we have that

A) M is conformally a punctured compact surface $\Sigma \sim \{p_1, \ldots, p_k\}$.

B) $\mathrm{Tan}(M, p)$ extends continuously (meromorphically, if $N = 3$) to Σ.

C)

$$-\int K dA = \int |K| dA = \begin{cases} 4\pi j & (R^3) \\ 2\pi j & (R^N) \end{cases}.$$

The discretation of total curvature is a very powerful fact which we can use to prove various regularity results. The basic idea is that any singularity (which is necessariy non-planar, by Allard) must "eat up" at least 4π worth of curvature.

Theorem 4.9. *Take $M^2 \subset R^3$ orientatable, minimally immersed. Suppose*

$$\frac{1}{2} \int |A|^2 = \int |K| dA \leq 4\pi - \epsilon.$$

Then

$$|A(M, \cdot)| \mathrm{dist}(\cdot, \partial M) \leq C_\epsilon < \infty.$$

The same statement is true in R^N with 2π instead of 4π.

Proof. Suppose not. Then we have a sequence $p_i \in M_i$ so that

$$|A(M_i, p_i)| \mathrm{dist}(p_i, \partial M_i) \to \infty,$$

but $\int_{M_i} |K| dA \leq 4\pi - \epsilon$.

Let q_i maximize

$$|A_i| \mathrm{dist}(\cdot, \partial M_i).$$

Set $M_i' = |A(M_i, q_i)|(M_i - q_i)$. Then $|A(M_i', 0)| = 1$ and

$$\mathrm{dist}(0, \partial M_i') \geq |A_i(p_i)| \mathrm{dist}(p_i, \partial M_i) \to \infty.$$

Now for any $x \in B_R(0)$ we have

$$|A_i'(x)| \leq \frac{\mathrm{dist}(0, \partial M_i')}{\mathrm{dist}(x, \partial M_i')} \leq \frac{\mathrm{dist}(0, \partial M_i')}{\mathrm{dist}(0, \partial M_i') - |x|} \to 1.$$

So we have uniform local curvature bounds on M_i'. Without loss of generality we can suppose M_i' is a single-sheeted graph over B_R (for $i >> 1$), since by throwing away other sheets we can only decrease the total curvature.

Then (subseq) $M_i' \to M$ in C^∞, where M is a smooth minimal surface in R^3, having $\partial M = 0$, $|A(M, 0)| = 1$, and

$$\int_M |K| < 4\pi.$$

Therefore by the tot-curv theorem M must be a plane, but this is a contradiction. $\square$

Remark 4.10. The only minimal surfaces with $\int |K| = 4\pi$ in R^3 are the catenoid and enneper's surface. To see this, observe that the Gauss map covers S^2 precisely once, and hence one can parameterize the surface (minus a point) by $S^2 - \{p\} = C$.

In general, if we have total curvature $< 4\pi k$, then we have curvature bounds away from the boundary and k points.

Theorem 4.11. *Let M_i^2 be a sequence of minimal surfaces in Ω^N, so that $\partial M_i \subset \partial\Omega$. Suppose*

$$\sup_i \frac{1}{2}\int_{M_i} |A|^2 \leq S < \infty.$$

Then there is a finite set X, with $\#X \leq \frac{S}{2\pi}$ (or 4π if $N = 3$), so that (subseq) $|A(M_i, \cdot)|$ is uniformly bounded on compact subsets of $\Omega \sim X$.

Proof. Define the measures on Ω

$$\beta_i(U) = \frac{1}{2}\int_{M_i \cap U} |A|^2.$$

By assumption the β_i are uniformly bounded, so we have (subseq) $\beta_i \rightharpoonup \beta$. Let

$$X = \{p \in \Omega : \beta(p) \geq 2\pi\} \quad \text{(or } 4\pi \text{ if } N = 3).$$

Trivially X contains at most $\beta(\Omega)/2\pi \leq S/2\pi$ points.

Given $p \in \Omega \sim X$, we know $\beta(p) < 2\pi - \epsilon$. So we can find an $r > 0$ so $\beta(B_r(p)) < 2\pi - \epsilon$ also. Hence $\beta_i(B_r(p)) < 2\pi - \epsilon$ for i sufficiently large, and hence $|A(M_i, \cdot)|$ is uniformly bounded on $B_{r/2}(p)$ by the previous theorem. $\square$

Here is another example of a planar characterization, and how you can get curvature bounds from it.

Theorem 4.12. *Suppose $M^2 \subset R^3$ is minimal, properly embedded, simply-connected, and has quadratic area growth. Then M is a flat plane.*

More generally one can replace "simply-connected" with "finite genus + 1 end".

Proof. Since M is simply-connected, we can have only one end. By theorem genus-tot-curv-equiv, we know M is complete with finite total curvature. So (after rotation), for any ϵ we can find an R so that $M \sim (B_R^2 \times R)$ is a graph over R^2 with slope $\leq \epsilon$.

So $\Theta(M, \infty) = 1$, but at any point $p \in M$ we have $\Theta(M, p) = 1$ also. So by monotonicity $M = \mathrm{Tan}(M, p)$ is a plane. $\square$

Theorem 4.13. *Take $M_i \subset \Omega \subset R^3$, with $\partial M_i \subset \partial\Omega$, a sequence of embedded minimal disks, such that*

$$\sup_i |M_i| < \infty.$$

Then curvature is uniformly bounded on compact sets, and hence (subseq) $M_i \to M$ smoothly.

We first need a lemma.

Lemma 4.14. *Let M be a minimal disk in R^3. Take B a ball, so that $\partial M \cap B = \emptyset$. Then $M \cap B$ is a union of disks.*

Proof. Let $C \subset M \cap B$ be a simple closed curve. We know $C = \partial D$ for some disk $D \subset M$. By the maximum principle, we must have $D \subset B$. □

proof of theorem. Suppose curvature blows up at $p \in \Omega$. Then (subseq) there are $p_i \to p$, $\lambda_i \to \infty$, so that $\lambda_i(M - p_i) \to M'$ smoothly. Here M' is a non-flat, smooth minimal surface in R^3 with quadratic area growth.

By the previous lemma M' is simply-connected. To see this, take $C \subset M'$ a simple closed curev. We can pick a sequence of curves $C_i \subset M_i$ with $C_i \to C$. But then the associated disks D_i converge to a disk $D \subset M$, $\partial D = C$.

I claim that M' is also embedded, possibly with multiplicity. By smooth convergence M' cannot have transverse intersections, and the multiplicity is finite since the areas of M_i are uniformly bounded.

Therefore by the previous theorem M' is a plane, a contradiction. □

Remark 4.15. The previous theorem would also work if Ω was a geodeisc ball of radius R in some 3-manifold, provided any geodesic ball of radius $\leq R$ has a smooth convex boundary.

5 removal of singularities

Throughout this section, let $M^2 \subset B_1(0) \sim \{0\} \subset R^n$ be a proper, branched minimal surface, so that $\partial M \subset \partial B_1$ and $\overline{M} \ni 0$.

Proposition 5.1. *We have $|M| < \infty$.*

Proof. Write $M(r, R) = M \cap (B_R \sim B_r)$, and choose the radial vector field $X(x) = x$. By minimality, on M we have

$$2 = div_M X = div_M X^T.$$

We calculate

$$2|M(r, R)| = \int_{\partial M(r,R)} X \cdot \nu \leq \int_{M \cap \partial B_R} X \cdot \nu \leq R|M \cap \partial B_R|,$$

since $X \cdot \nu \leq 0$ on $M \cap \partial B_r$. □

In fact this result holds if M is only a minimal varifold, by plugging a test function like $X(x) = \phi(|x|)\eta(|x|)x$. Here ϕ approximates $1_{[r,\infty)}$ and η approximates $1_{[0,1/2)}$.

Corollary 5.2. *Monotonicity still holds at* 0*. In particular,* $\Theta(M, 0) < \infty$*.*

Proof. Direct from the inequality $2|M \cap B_R| \leq R|M \cap \partial B_R|$. □

Proposition 5.3. *M has finitely many ends at* 0*, in the sense that*

$$\limsup_{r\to 0}(\#\ \textit{components of}\ M \cap B_r) < \infty.$$

Proof. Let E be a component of $M \cap B_{2r}$. Choose $x \in E \cap \partial B_r$, and we have

$$1 \leq \frac{|M \cap B_r(x)|}{\pi r^2} \leq 4\frac{|M \cap B_{2r}|}{\pi(2r)^2}.$$

Therefore #ends $\leq 4\Theta(M, 0) < \infty$. □

Theorem 5.4. *If M has finite genus, then $M \cup \{0\}$ is a branched minimal surface.*

Proof. Since #ends $= E < \infty$, we know M is homeomorphic to a compact manifold with boundary, minus E points. Each end corresponds (topologically) to a punctured disk.

Conformally, each end is either $D^2 \sim \{0\}$, or an annulus $A(\rho, 1) \subset D^2$. If the former, then the a.c. immersion $F : D^2 \sim \{0\} \to R^n$ is a bounded harmonic function, and therefore extends across 0.

(recall a.c. means: $\Delta_{eucl}F = 0$, and $|F_x| = |F_y|$, $F_x \cdot F_y = 0$).

If the latter, then we can extend the immersion $F : A(\rho, 1) \to R^n$ to a bigger annulus $A(\rho^2, 1)$ by Schwarz reflection. The extension is still almost-conformal, but now has an entire circle ∂B_ρ of singularities. This is a contradiction, since a.c. maps can have at most isolated singularities. □

6 Gauss-Bonnet and branch points

Take M a branched minimal surface, with smooth boundary ∂M free of branch points.

Observation 6.1. *We have*

$$-2\pi \sum_{\textit{branch points}} Ord(p) + \int_M K dA + \int_{\partial M} k ds = 2\pi\chi(M).$$

(A branch point has order m if it is locally parameterized by $z \mapsto z^{m+1}$.)

Proof. Remove a little ball around each branch point. For each ball we introduce a new boundary, which is approximately approximately a circle winding $Ord(p)$-times around. For simplicity suppose we have just one branch point of order m, and write B for the ball we remove.

We have

$$\int_{M\sim B} KdA + \int_{\partial M} kds - 2\pi m = 2\pi\chi(M \sim B) = 2\pi(\chi M - 1).$$

□

7 main theorem

We show that for 2 dimensional minimal surfaces, either control of total curvature or genus will give compactness.

Lemma 7.1. *Let M^2 and N^{n-1} be transversely intersecting manifolds, and let γ be the curve of intersection. If k is the ambient curvature of γ, then*

$$|k| \leq \frac{|A_M(\gamma', \gamma')| + |A_N(\gamma', \gamma')|}{\sin\alpha}.$$

Proof. We first prove the following claim: let n_1 and n_2 be unit vectors in R^2, separated by non-zero angle $\alpha \in (0, \pi/2]$. Then for any vector v,

$$|v| \leq \frac{|v \cdot n_1| + |v \cdot n_2|}{\sin\alpha}.$$

We have

$$v \cdot n_1 = |v|\cos\beta_1, \quad v \cdot n_2 = |v|\cos\beta_2,$$

where $|\beta_1 - \beta_2| = \alpha$. So for any λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 = 1$,

$$|v| = \frac{\lambda_1}{|\cos\beta_1|}|v \cdot n_1| + \frac{\lambda_2}{|\cos\beta_2|}|v \cdot n_2|.$$

Choose λ_i so that $\frac{\lambda_1}{|\cos\beta_1|} = \frac{\lambda_2}{|\cos\beta_2|}$, and we have

$$|v| = \frac{|v \cdot n_1| + |v \cdot n_2|}{|\cos\beta_1| + |\cos\beta_2|}.$$

It will thus suffice to show that,

$$|\cos(x+\alpha)| + |\cos x| \geq \sin\alpha.$$

Since the LHS is minimized when the two are equal, one can clearly see this is satisfied since

$$\frac{2\cos(\pi/2 - \alpha/2)}{\sin\alpha} = \frac{1}{\cos(\alpha/2)} \geq 1.$$

This proves the claim.

Now let $k = \nabla_{\gamma'}\gamma'$ be the curvature vector. Recall that $A_M(\gamma', \gamma')$ is the projection of k onto the normal bundle $N(M)$ of M, and similarly for A_N.

Since $k \in N(M) \cap N(N)$, by transversality and dimensions we have

$$k \in \text{span}\{A_M(\gamma', \gamma'), A_N(\gamma', \gamma')\} = R^2.$$

Therefore the result follows by the claim. □

Theorem 7.2. *Let $M_i \subset \Omega \subset$ (n-manifold) be a sequence of minimal surfaces, with $\partial M_i \subset \partial\Omega$.*

*Suppose for all $K \subset\subset \Omega$ we have $\sup_i |M \cap K| < \infty$. Further, suppose *one* of the following:*

1): $\sup_i$ tot curv$(M_i \cap K) < \infty$ (counting possible branch points), for all $K \subset\subset \Omega$.

2): $\sup_i \text{genus}(M_i \cap U) < \infty$, for all U so that $\overline{U} \subset\subset \Omega$.

Then (subseq) $M_i \to M$, where M is a branched minimal surface, and convergence is smooth away from some isolated points.

Proof. We shall prove that conditions 1) and 2) are equivalent. Assume, for the moment, this is the case. Then 1) implies uniform curvature bounds away from a discrete set X (theorem tot-curv-implies-bounds), hence $M_i \to M$ smoothly away from a X. Since M has locally-finite genus, and local area bounds, the singularity removal theorem says M extends across X as a branched minimal surface.

The equivalence 1) $\equiv$ 2) follows from a beautiful "local Gauss-Bonnet" due to Ilmanen, but we will prove a simplified version.

We first prove 2) $\implies$ 1). Since the result is local we can assume $\Omega = B_5(0) \subset R^n$ with some metric, genus$M_i \leq g < \infty$, $|M_i| \leq A < \infty$, and that all balls in Ω are convex. We work towards uniformly bounding $\int_{B_1 \cap M_i} |A|^2$.

Take $M = M_i$. Let $r \in (2, 3)$. If C is a component of $M \sim B_r$, then by the maximum principle there must be a point $x \in C \cap \partial B_4$. Therefore by monotonicity we have $|C| \geq |C \cap B_1(x)| \geq a(\Omega)$.

Now if we remove a curve from $M \cap \partial B_r$, then we can only add a component or decrease the genus. So

$$\#(M \cap \partial B_r) \leq g + A/a.$$

Let $\phi = \phi(|x|)$ be a cutoff function, supported in $[0, 3)$, equal to 1 on $[0, 2]$, and with $\frac{|\phi'|^2}{\phi} \leq 2$. Given a curve $\gamma \subset M$, let $\tilde{k}$ be the curvature of γ in M, and k be the curvature in R^n. Using Gauss-Bonnet, we obtain

$$\begin{aligned}
\int_M \phi|A|^2 dA &= \int_0^1 \int_{M\cap\{\phi>t\}} |A|^2 dA dt \\
&\leq cA + \int_0^1 \left(2\int_{M\cap\{\phi=t\}} \tilde{k} ds - 8\pi\#(M \cap \{\phi > t\}) + 8\pi g + 2\pi\#(M \cap \{\phi = t\}) \right) dt \\
&\leq cA + 10\pi g + A/a + 2\int_0^1 \int_{\partial M\cap\{\phi=t\}} |k| ds dt.
\end{aligned}$$

We used that $|A|^2 = -2K + O(\Omega)$.

We now bound the last integral. Using lemma bound-curve-curvature with M and ∂B_r, we have

$$|k| \leq \frac{|A| + c/r}{\sin\alpha}.$$

We also know that $|\nabla^M r| = \sin\alpha$. By the coarea formula we have

$$\begin{aligned}
\text{(last integral)} &\leq 2\int_M \phi' \sin\alpha |k| dA \\
&\leq 2\int_M \phi'(|A| + c/r)dA \\
&\leq 2A + \int_M \frac{|\phi'|^2}{\epsilon\phi} + \epsilon\phi|A|^2 + \phi' c/r dA \\
&\leq cA + \epsilon\int_M \phi|A|^2 dA + c\int_{M\cap B_3} 1/r dA.
\end{aligned}$$

Using monotonicity the very last integral is bounded like

$$\begin{aligned}
\int_{M\cap B_3} 1/r dA &= \int_0^\infty \mathcal{H}^2(M \cap B_3 \cap \{1/r > t\})dt \\
&= \int_{1/3}^\infty \mathcal{H}^2(M \cap B_{1/t})dt \\
&\leq A\int_{1/3}^\infty 1/t^2 dt \\
&\leq 3A.
\end{aligned}$$

We deduce that

$$\int_{M\cap B_1} |A|^2 \leq cA + 10\pi g.$$

Let us now prove 1) $\implies$ 2). Consider as before $\Omega = B_5 \subset R^n$ with some metric, $M \subset \Omega$, $\partial M \subset \partial\Omega$, and we have uniform area bounds $|M| \leq A$. Consequently, the number of components is bounded by A/a.

Taking ϕ as above, we have

$$\begin{aligned}
\int_M \phi|A|^2 dA &\geq -cA + \int_0^1 \left(2\int_{M\cap\{\phi=t\}} \tilde{k} ds - 8\pi\#(M \cap \{\phi > t\}) + 8\pi g(M \cap \{\phi > t\})\right) dt \\
&\geq -cA - 2\int_0^1 \int_{M\cap\{\phi=t\}} |\tilde{k}| ds dt + 8\pi\int_0^1 g(M \cap \{\phi > t\}).
\end{aligned}$$

And therefore, using the same inequalities as before, we have

$$\int_0^1 g(M \cap \{\phi > t\}) \leq cA + c\int_{M\cap B_3} |A|^2.$$

Consequently, since genus is monotone in t, we deduce

$$g(M \cap B_{1/2}) \leq cA + c\int_{M\cap B_3} |A|^2. \qquad \square$$

8 PDEs

Suppose $F : R^n \to R^k$ is smooth, non-linear, and $F(x) = 0 = F(y)$. Then $M(x-y) = 0$ for some "interesting" matrix. Precisely, we have

$$\begin{aligned} F(y) - F(x) &= \int_0^1 \frac{d}{dt} F(x + t(y-x))dt \\ &= \left(\int_0^1 DF(x + t(y-x))dt \right)(y-x). \end{aligned}$$

This basic logic underlies the following crucial fact: the difference of two solutions to a non-linear PDE satisfies a linear PDE.

Theorem 8.1. *Let*

$$L[u] = a_{ij}(x, u, Du)D_{ij}u + b_i(x, u)D_i u + c(x, u)u + f(x, u).$$

Then

$$L[u] - L[v] = \tilde{L}[u - v]$$

where $\tilde{L}$ is a linear, homogenous PDE.

We apply this to minimal surfaces. Let $M^n \subset R^{n+1}$ be minimal. Locally we can write M as a graph $u : R^n \to R$, satisfying

$$\begin{aligned} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) &= 0 \\ &= \left(\frac{\delta_{ij}}{\sqrt{1 + |Du|^2}} - \frac{D_i u D_j u}{(1 + |Du|^2)^{3/2}} \right) D_{ij} u \\ &= a_{ij}(Du)D_{ij}u. \end{aligned}$$

(the above PDE is called the minimal surface equation (MSE))

Taking this as our $L[u]$, we have

$$\begin{aligned} L[u] - L[v] &= a_{ij}(Du)D_{ij}(u - v) + (a_{ij}(Du) - a_{ij}(Dv))D_{ij}v \\ &= a_{ij}(Du)D_{ij}\phi + b_k(Du, Dv, D^2 v)D_k \phi \end{aligned}$$

where $\phi = u - v$. In the last equality we used principle EQREF again. Notice this is linear because we *freeze* the coefficients at u, v, and view them as functions of x only.

Now suppose $u_p, v_p \to w$ nicely, with u_p and v_p both solutions to the MSE for every p. Then

$$\tilde{L}_p \phi_p = a_{ij}(Du_p)D_{ij}\phi_p + b_k(Du_p, Dv_p D^2 v_p)D_k \phi_p$$

and the coefficients converge nicely to $a_{ij}(Dw)$, $b_k(Dw, D^2 w)$.

I claim that $\tilde{L}_p \to \tilde{L}$, the linearization of the MSE at w. We have

$$\tilde{L}_p[\phi_p].....$$

Theorem 8.2. *Take M, N two minimal hypersurfaces in Ω, with M dividing Ω into two components U, V. Suppose $N \subset \bar{U}$. Then either N, M are disjoint, or they coincide.*

Proof. Suppose M, N touch at p. Of course they touch tangentially by our hypothesis. Write both as graphs $u, v : B^n \to R$ over T_pM. Then the difference $u - v \geq 0$ satisfies a homogeneous, elliptic, linear PDE. By the maximum principle they must coincide. □

MORE STUFF...

9 second variation

Let $F_t : M \to R^N$ be a family of immersions. Then of course

$$area(F_t) = \int_M \sqrt{|DF^T DF|}.$$

We've seen that first-order minimality is characterized by $H = 0$. We now look to the second variation of area.

Fix M minimal surface in R^N, and let $F_{x,y} : M \to R^N$ be a two-paramter family of deformations, with $F_{0,0} = Id$. Write $M_{x,y} = F_{x,y}(M)$. We calculate

$$\frac{\partial^2 A}{\partial x \partial y}|_{0,0} = -\int < DH, \partial_x F > \partial_y F dA.$$

Here $J = DH$ is a second order differential operator (the "Jacobi operator"). Since $\partial_x \partial_y A = \partial_y \partial_x A$, J is self-adjoint.

Restrict to M being 2-sided. Then $X = \partial_x F = f\nu$, and $Y = \partial_y F = g\nu$. We have

$$\partial_x \partial_y A|_{0,0} = \int (-\Delta f - Ric(\nu, \nu) f - |A|^2 f) g dA = \int J(f) g dA.$$

Observe that $J(f) = 0$ is the linearization of the minimal surface equation $H = 0$.

Definition 9.0.1. M is stable $\iff$ $\partial_x^2 A \geq 0$ for all compactly supported variations $\iff$ $\int J(f) f \geq 0$ for all f.

Theorem 9.1. *Suppose M is closed, stable, 2-sided minimal in N, with $Ric \geq 0$. Then $|A| \equiv 0$ and $\mathrm{Ric}(\nu, \nu) \equiv 0$.*

Corollary 9.2. *If N has $Ric > 0$, then $\not\exists$ closed, stable, 2-sided minimal hypersurfaces.*

Take $M^2 \subset R^3$.

Theorem 9.3 (Fischer-Colbrie-Schoen)**.** *If M complete, orientable, stable, then M is a plane.*

Corollary 9.4. *If M has boundary, then $|A|\text{dist}_M(\cdot, \partial M) \leq C < \infty$.*

Theorem 9.5. *Let M_i, M_i' be disjoint, minimal hypersurfaces. Suppose $M_i, M_i' \to M$ smoothly. Then M is stable.*

Proof. Write M_i, M_i' as normal graphs $u_i, u_i' : M \to R$. Then $u_i - u_i' > 0$, and $L_{u_i,u_i'}(u_i - u_i') = 0$. Since $u_i, u_i' \to 0$, we have $L_{u_i,u_i'} \to J =$ the Jacobi operator on $M =$ linearization of L at $(0, 0)$.

Choose $p \in M$, and set $v_i = \frac{u_i - u_i'}{u_i(p) - u_i'(p)}$. Then $L_{u_i,u_i'}(v_i) = 0$, $v_i(p) = 1$, and $v_i > 0$.

By Harnack we have v_i is uniformly bounded away from $0, \infty$ on compact sets. Therefore by Schauder we have (for a compact sets $K \subset\subset \hat{K}$)

$$||v_i||_{2,\alpha;K} \leq C_i||v_i||_{0;\hat{K}} \leq C_{\hat{K}}$$

So after taking a subsequence we have $v_i \to v$ a positive Jacobi field on M.

Let's see how this implies stability. Suppose M is unstable. Then on some compact $\Omega \subset M$ we have $\lambda_1(\Omega) < 0$. Take $\phi_1 \geq 0$ the first eigenfunction. (since J is self-adjoint, elliptic, and 2nd order linear, we have a ON eigenbasis) Of course $\phi_1|_{\partial\Omega} = 0$, and Ω is bounded, and so we can rescale ϕ_1 until $\phi_1 < v$. Now scale back up until we touch, and we violate the maximum principle. □

Remark 9.6. M is stable $\iff$ $M \sim \{p_1, \ldots, p_k\}$ is stable. More generally, by the Raleigh quotient we have for any eigenvalue $\lambda_k(M) = \lambda_k(M \sim \{p_1, \ldots, p_k\})$.

Recall the previous theorem:

Theorem 9.7. *Take $M_i^2 \subset \Omega$ a sequence of embedded, oriented, minimal surfaces, with $\partial M_i \subset \partial\Omega$. Suppose area and genus are locally uniformly bounded area, genus.*

Then (subseq) $M_i \to M$, where M is minimal, smooth, embedded, possibly with multiplicity, and convergence is smooth outside a discrete set Q.

We can refine our information about the limit surface M.

Theorem 9.8. *If Σ is a component of M with multiplicity > 1, then Σ is stable.*

Proof. Since the convergence to $\Sigma \sim Q$ is smooth, the limiting surfaces are disjoint, and therefore by the previous theorem $\Sigma \sim Q$ must be stable. But then this implies Σ is stable also. □

Corollary 9.9. *If $\hat{\Sigma}$ is an unstable component, then $\hat{\Sigma} \cap Q = \emptyset$.*

We can now refine our information on blow-ups.

Theorem 9.10. *Take $M_i^2 \subset \Omega^3$, $\partial M_i \subset \partial\Omega$ minimal. Suppose curvature blows up at p. Then (we already know) for some sequence $p_i \to p$ and $\lambda_i \to \infty$, we have (subseq) $\lambda_i(M_i - p_i) \to M$ smoothly, where M is smooth, non-flat.*

The new observation is that M must have multiplicity 1.

10 getting local area bounds

We know that locally uniformly bounded area and genus gives compactness (up to multiplicity and away from a discrete set). Often we impose genus bounds, but have no a priori control over area. Here we discuss a way to obtain uniform local area bounds.

Let $M_i \subset \Omega$ be minimal submanifolds (or indeed varifolds).

Definition 10.0.1. The area blow-up set Z is the (closed) set

$$Z = \{x \in \Omega : \limsup |M_i \cap B(x,r)| = \infty, \text{ for all } r > 0\}.$$

Definition 10.0.2. A closed subset $Z \subset \Omega$ is an (m,h)-set if: for every smooth $f : \Omega \to R$, so that $f|_Z$ has a local max at $p \in Z$, we have that

$$\text{trace}_m(D^2 f(p)) \le h|Df(p)|.$$

Here trace_m is the sum of the lowest m eigenvalues.

The idea is that (m,h) sets represent a set-theoretic version of an "m-dim manifold with ∂, having $|H| \le h$". In particular, if M^m is a submanifold with $M \subset \Omega$, $\partial M \subset \partial\Omega$, then M is an (m,h)-set $\iff$ $|H| \le h$.

Theorem 10.1 (main theorem)**.** *Suppose $M_i \subset \Omega$ are m-dim submanifolds, with $|H_{M_i}| \le h$, and $\sup |\partial M_i|(U) < \infty$ for all $U \subset\subset \Omega$.*

Then the area blow-up set Z is an (m,h)-set.

Remark 10.2. 1. Z is an (m,h)-set $\iff$ μZ is an $(m, h/\mu)$-set.

2. If $Z_i \to Z$ (in the Hausdorff sense), with each Z_i an (m,h)-set, then Z is an (m,h)-set also.

3. The set of numbers $\{h : Z = (m,h) \text{ set}\}$ is closed.

Theorem 10.3 (constancy theorem)**.** *Suppose $Z \subset$ a proper, connected m submanifold $M \subset \Omega$, and Z is an (m,h)-subset of Ω.*

Then either $Z = \emptyset$ or $Z = M$.

In particular, the above two theorems imply the following corollary.

Corollary 10.4. *If a sequence of minimal surfaces converge (as sets) to a subset of a smooth surface, then the area blow-up set is either nowhere or everywhere. Alternatively, if in this scenario we have multiplicity 1 convergence somewhere, then we must have smooth convergence everywhere.*

main theorem. Suppose not. Then there is a smooth $f : \Omega \to R$ such that $f|_Z$ has a local max at p, but $\text{trace}_m D^2 f(p) - h|Df(p)| > 0$. WLOG we can suppose $\Omega = B_1$, $p = 0$, $f|_Z$ has a strict local max at 0, and $\{f \ge t\}$ is compact for all t. To achieve the last two reductions one would replace f with a function like

$$f - \frac{|x|^4}{1 - |x|^2}.$$

On a small ball B we have $\text{trace}_m D^2 f - h|Df| \geq \delta > 0$. By adding a constant, we can assume $\max_{Z \sim \mathring{B}} f < 0 < f(0)$.

Let $N = \{f \geq 0\}$ (compact), and WLOG let 0 be a regular value of f, so in particular ∂N is smooth. Then since $N \sim \mathring{B}$ is compact, disjoint from Z, we have

$$|M_i|(N \sim \mathring{B}) \leq A < \infty.$$

Of course we also have $\max_N |f||Df| \leq \Gamma < \infty$, $\min_B f \geq \gamma$, $\min_N f \text{trace}_m D^2 f > -\tau$.

Let $X = \nabla(f^2/2) = f\nabla f = fD^T f$. Then $|X| \leq \Gamma$ on N, and we have

$$DX = fD^2 f + D^T f^T Df \geq fD^2 f,$$

so $\text{trace}_m DX \geq \text{trace}_m fD^2 f = f\text{trace}_m D^2 f$ (using that $f \geq 0$ on N).

Now we have

$$\begin{aligned}
\int_{M_i \cap N} div_{M_i} X dA &= -\int_{M_i \cap N} H_i \cdot X dA + \int_{\partial(M_i \cap N)} X \cdot \nu ds \\
&\leq \int_{M_i \cap N} h|X| dA + \int_{\partial(M_i \cap N)} |X| ds \\
&= \int_{M_i \cap N} h|X| dA + \int_{\partial M_i \cap N} |X| ds \\
&\leq \int_{M_i \cap N} h|X| dA + \Gamma |\partial M_i|(N).
\end{aligned}$$

By assumption the right-most term is $O(1)$.

So we have, taking $0 \in B^*$ to be some small ball in the interior of $N \cap B$,

$$\begin{aligned}
&(\min_{B^*} f)\delta |M_i|(B^*) \leq \\
&\int_{M_i \cap N \cap B} f(\text{trace}_m D^2 f - h|Df|) dA \leq \\
&\int_{M_i \cap N \cap B} \text{trace}_m DX - h|X| dA \leq \\
&\int_{M_i \cap N \cap B} div_{M_i} X - h|X| dA \leq \int_{(M_i \cap N) \sim B} h|X| - div_{M_i} X dA + O(1) \\
&\leq C|M_i|(N \sim B) + O(1) \\
&\leq CA + O(1)
\end{aligned}$$

Therefore the above calculation shows that $|M_i|(B^*) \leq O(1)$. But by assumption $0 \in B^*$ is in the area blow-up set Z. This is a contradiction. □

Theorem 10.5. *Suppose $M \subset \Omega$ is an m-dim manifold, with $\partial M \subset \partial \Omega$ and $|H| \leq h$. Then M is an (m, h)-set.*

Proof. Set $M_k = kM$ (i.e. M with multiplicity k). Then the area blow-up set is precisely M, and the result follows by the previous characterization. □

Theorem 10.6 (barrier principle)**.** *Let Z be an (m,h)-set in Ω, with $Z \subset$ a closed region N having smooth boundary ∂N. Suppose there is a $p \in Z \cap \partial N$.*

*Then $\sum_{i=1}^m \kappa_i(p) \leq h$, where $\kappa_1 \leq \ldots \leq \kappa_{n-1}$ are the principle curvatures of ∂N, with respect to the *inward* unit normal.*

Proof. Let $u : \Omega \to R$ be the signed distance function to ∂N, having $u > 0$ outside N. Set $f = e^{\alpha u}$ for some $\alpha > 0$. Then

$$Df = \alpha e^{\alpha u} Du, \quad D^2 f = \alpha^2 e^{\alpha u} Du^T Du + \alpha e^{\alpha u} D^2 u.$$

Choose a basis of $T_p\Omega$ so that $e_1, \ldots, e_{n-1}$ correspond to principle directions of ∂N. In this basis $D^2 u(p) = \mathrm{diag}(\kappa_1, \ldots, \kappa_{n-1}, 0)$, and so

$$D^2 f(p) = \mathrm{diag}(\alpha\kappa_1, \ldots, \alpha\kappa_{n-1}, \alpha^2) e^{\alpha u}.$$

By choosing α sufficiently large we can ensure $\mathrm{trace}_m D^2 f(p) = \sum_{i=1}^m \alpha\kappa_i$. Therefore, since by construction $f|_Z$ attains a local maximum at p, we have

$$\alpha \sum_{i=1}^m \kappa_i - h\alpha = \mathrm{trace}_m D^2 f(p) - h|Df(p)| \leq 0.$$

□

Proof of constancy theorem. Suppose not, so there is a $p \in M \sim Z$. Choose $q \in Z$ closest to p. Take a sequence of dilations about q, and obtain a (subsequential) limit $(m, 0)$-set $Z_\infty \subset R^m$. By construction $Z_\infty \subset \{x_1 \leq 0\} \times R^{m-1} \times \{0\}$.

Let $f = x_1 + x_1^2 + \sum_{i>m} x_i^2$. Then

$$D^2 f = \mathrm{diag}(2, 0, \ldots, 0, 2, \ldots 2)$$

(having $m-1$ zeros). In particular, $\mathrm{trace}_m D^2 f = 2 \not\leq 0$, and yet $f|_Z$ has a local maximum at 0. □

Here's how you can use these theorems to obtain compactness results for 2-dimensional minimal surfaces. Suppose you have a sqeuence of minimal surface M_i, with uniform control over $|\partial M_i|$ and genus(M_i). The area blow-up set Z is a $(2,0)$-set, and in particular any minimal surface will act as a barrier.

By a Meek-Hoffman-type theorem for $(2,0)$ sets ($\nexists$ djsinot, complete, properly immersed minimal surfaces in R^3), one can deduce that $Z = \emptyset$.