Math 60670 Homework 9

Due Wednesday April 22.

Problem 1: Let (M^n, g) be a Riemannian manifold, and $p \in M$, and x^i normal coordinates of M about p.

A. Show that the metric g_{ij} in these normal coordinates admits the Taylor expansion about 0:

$$g_{ij}|_x = \delta_{ij} - \frac{1}{3} R_{iklj}|_p x^k x^l + O(|x|^3). \tag{1}$$

Hint: Let $\gamma(t) = tv$ be a radial geodesic, and $J(t) = tW^i \partial_i$ be a Jacobi field along γ, and compute the first four t-derivatives of $|J(t)|^2$ at 0 in two ways.

B. Use this to show that the volume form has the expansion

$$dV|_{tv} = 1 - (t^2/6)\mathrm{Ric}|_p(v, v) + O(t^3)$$

and thereby deduce that the volume of a small geodesic ball $B_r(p) \subset M$ has the expansion

$$\mathrm{Vol}_g(B_r(p)) = \omega_n r^n \left(1 - \frac{r^2 \mathrm{Scal}(p)}{6(n+2)} + O(r^3) \right), \tag{2}$$

where ω_n is the Euclidean volume of the Euclidean unit ball.

C. Bonus: Compute one further term in expansion (1) to get that the error in (2) is in fact $O(r^4)$.

Problem 2: A Riemannian manifold (M^n, g) is called Einstein if $\mathrm{Ric}(X, Y) = \lambda g(X, Y)$ for some function $\lambda : M \to \mathbb{R}$.

A. Show that if M is Einstein and connected and $n \geq 3$, then λ is constant. Hint: In geodesic coordinates at p, note the second Bianchi identity is $\partial_s R_{ijkl}|_p + \partial_i R_{jskl}|_p + \partial_j R_{iskl}|_p = 0$, then take the trace twice.

B. Show that if M is Einstein and connected and $n = 3$, then M has constant sectional curvature.

Problem 3: Show that the paraboloid $\{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ is complete, non-compact, and has positive sectional curvature.