Math 60670 Homework 8

Due Wednesday, April 15.

Q1. Let (M^n, g) be a Riemannian manifold. A domain $U \subset M$ is called geodesically convex if: whenever a minimizing geodesic $\gamma : [0, \ell] \to M$ satisfies $\gamma(0), \gamma(\ell) \in U$, then $\gamma(t) \in U$ for all t.

A. Given any p, show there is a radius $r > 0$ so that the in $B_{3r}(p)$ square-distance function $f(x) := d(x,p)^2$ is smooth and convex in the sense that $\nabla^2 f > 0$. Hint: look at $\nabla^2 f$ at p in normal coordinates, or simply choose coordinates in which the metric becomes sufficiently C^1 close to Euclidean.

B. Show that the ball $B_r(p)$ is geodesically convex.

Q2. Fix (M^2, g) a compact Riemannian 2-manifold without boundary. Let us call $B_r(p)$ a uniformly normal neighborhood if for every $q \in B_r(p)$, the ball $B_r(q)$ is a normal neighborhood (i.e. $\exp_q$ is a diffeomorphism $B_r(0) \to B_r(q)$).

A. Suppose $B_r(p)$ is a uniformly normal neighborhood, and geodesically convex. Let $\Omega \subset B_{r/3}(p)$ be any (open) geodesically convex subset. If $q \in \Omega$, show that $\exp_q^{-1}(\Omega)$ is star-shaped, and hence Ω is contractable.

B. Given $B_r(p)$, Ω as in part A, suppose $\gamma : [0,1] \to B_{r/3}(p)$ is a geodesic "bisecting" Ω in the sense that $\gamma(0), \gamma(1) \in \overline{\Omega} \setminus \Omega$. Show that each connected component of $\Omega \setminus \gamma([0,1])$ is also geodesically convex.

C. Given $B_r(p)$ as in part A, take $x, y, z \in B_{r/3}(p)$, and show that the region bounded by the geodesics connecting x, y, z is a geodesically convex geodesic triangle. In a similar vein, show that one can find a geodesically convex geodesic polygon Ω satisfying $B_{r/4}(p) \subset \Omega \subset B_{r/3}(p)$. Hint: choose sufficiently many points along $\partial B_{r/3}(p)$, and then connect them with geodesics.

D. Show that one can find $r > 0$ and a finite collection of points $\{x_i\}_i \subset M$ so that each $B_r(x_i)$ is uniformly normal and geodesically convex, the balls $\{B_{r/4}(x_i)\}_i$ cover M.

E. Construct for each x_i a geodesically convex geodesic polygon Ω_i as in part C so that $B_{r/4}(x_i) \subset \Omega \subset B_{r/3}(x_i)$. Argue that by adding in vertices and geodesic edges to the collection of polygons $\{\Omega_i\}_i$, one may subdivide $\{\Omega_i\}_i$ into a triangulation of M by geodesic triangles.