Q1 A. $f(x) = |\exp_p^{-1}(x)|^2$

$$\Rightarrow \partial_i f\Big|_x = 2\langle D\exp_p^{-1}\big|_x(\partial_i),\ \exp_p^{-1}(x)\rangle$$

$$\Rightarrow \partial_i\partial_j f\Big|_p = 2\langle D^2\exp_p^{-1}\big|_p(\partial_i,\partial_j),\ \overbrace{\exp_p^{-1}(p)}^{=0}\rangle + 2\langle D\exp_p^{-1}\big|_p(\partial_i),\ \underbrace{D\exp_p^{-1}\big|_p}_{=Id}(\partial_j)\rangle$$

$$= 0 + 2\langle\partial_i,\partial_j\rangle$$

$$= 2\delta_{ij}$$

since $\Gamma_{ij}^k\big|_p = 0$

$$\Rightarrow \nabla^2 f\big|_p(\partial_i,\partial_j) = \partial_i\partial_j f\big|_p = 2\delta_{ij} > 0$$

$\Rightarrow \exists\, r>0$ st $\nabla^2 f\big|_x > 0$ for $x \in B_{3r}(p)$

by continuity of $x \mapsto \nabla^2 f\big|_x$

B. spse $q, q' \in B_r(p)$ and $\gamma : [0,\ell] \to M$ = min'z geodesic $q \to q'$

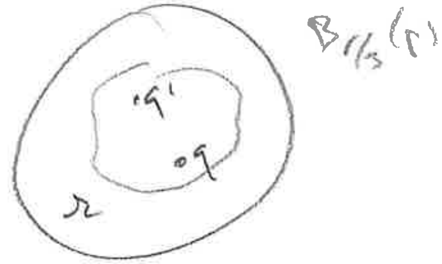
$\Rightarrow L\gamma < 2r$

$\Rightarrow d(p,\gamma(t)) < 3r \quad \forall t$

$$\Rightarrow \frac{d^2}{dt^2} f(\gamma(t)) = \gamma'(\gamma'(f))\big|_{\gamma(t)} = \gamma'(\gamma'(f))\big|_{\gamma(t)} - \overbrace{(\nabla_{\gamma'}\gamma')}^{\text{since geodesic}}(f)\big|_{\gamma(t)}$$

$$= \nabla^2 f\big|_{\gamma(t)}(\gamma',\gamma') > 0 \Rightarrow d(p,\gamma(t)) \le \underbrace{\max(d(p,\gamma(0)),\ d(p,\gamma(\ell)))}_{< r}$$

Q2. A.



take $q, q' \in \Omega$

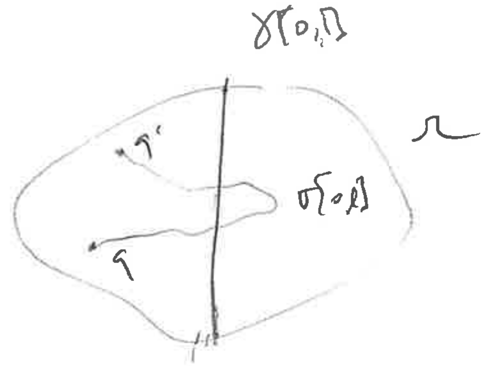
since Ω = convex, and $B_r(q)$ = normal neighborhood

$\Rightarrow \gamma(t) = \exp_q(t \exp_q^{-1}(q')) \in \Omega \quad \forall t \in [0,1]$

$\Rightarrow t \exp_q^{-1}(q') \in \exp_q^{-1}(\Omega) \quad \forall t \in [0,1]$

$\Rightarrow \exp_q^{-1}(\Omega)$ = star shaped

B.



let Ω' = connected cpnt of $\Omega \setminus \gamma[0,1]$

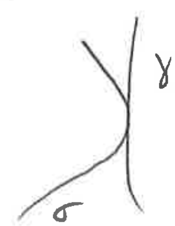
$q, q' \in \Omega'$

$\sigma : [0,\ell] \to \Omega$

= min'z geodesic $q \to q'$

if $\sigma[0,\ell] \not\subset \Omega' \Rightarrow \sigma$ must intersect γ

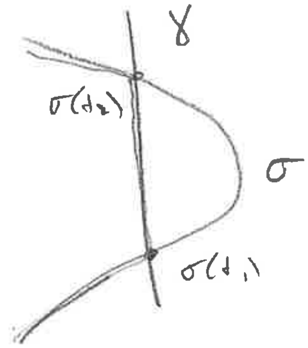
↳ if σ meets γ tangentially



$\Rightarrow \sigma$ coincides w/ γ by uniqueness of geodesics

but $q, q' \notin \gamma[0,1]$ ↯

↳ if σ crosses γ tranversely

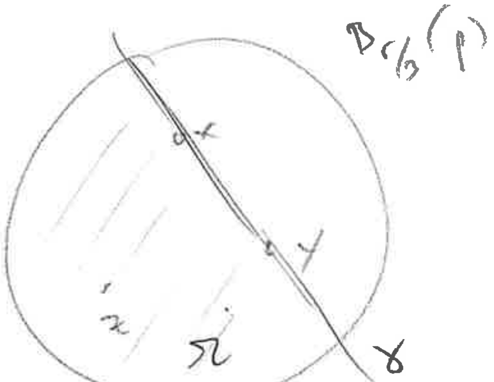


$\Rightarrow$ replace $\sigma|_{[t_1, t_2]}$ with γ

$\Rightarrow$ still minz since γ minz

but new curve not smooth ↯

C



let $\gamma = \overset{\text{minz}}{\text{geodesic}}\ x \to y$

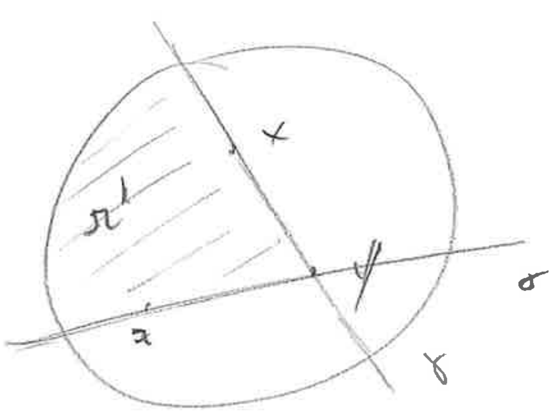
↳ extent γ to exist on entire $B_{r/3}(p)$

$\Rightarrow$ γ = minz on $B_{r/3}(p)$

since $B_r(x)$ = normal

$\Rightarrow$ γ bisects $B_{r/3}(p)$ into convex connected components (by B)

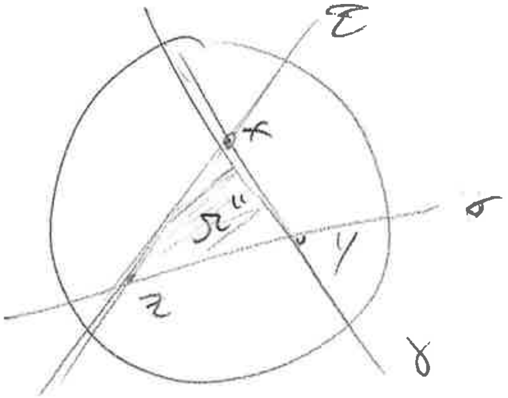
$\Rightarrow$ component Ω of $B_{r/3}(p) \setminus \gamma$ containing z = convex



do the same with geodesic σ connecting z to y

$\Rightarrow$ σ bisects Ω into convex components

$\Rightarrow$ component Ω' of $\Omega \setminus \sigma$ meeting x is convex



repeat with geodesic τ connecting $z \to x$

$\Rightarrow$ left w/ convex geodesic triangle Ω'' w/ vertices x, y, z

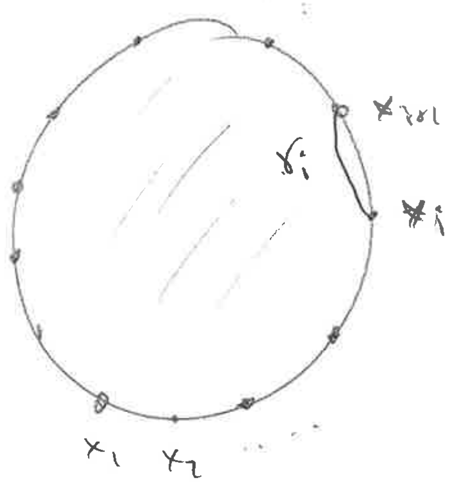
choose $\{x_i\} \subset \partial B_{r/3}(p)$ a maximally ρ-separated set

$\hookrightarrow$ if we order $\{x_i\}$ by the counterclockwise rotation around $\partial B_{r/3}(p)$

then $d(x_i, x_{i+1}) < 2\rho$

$\Rightarrow$ min'z geodesic γ_i connecting $x_i \to x_{i+1}$ lies in $B_{2\rho}(x_i)$

$\subset 2\rho$ - neighborhood of $\partial B_{r/3}(p)$



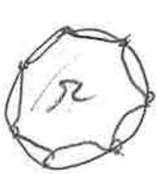
ensure $2\rho < r/3 - r/4$

$\Rightarrow \gamma_i$ lies outside $B_{r/4}(p)$

and since γ_i bisects $B_{r/3}(p)$ = convex

$\Rightarrow$ component of $B_{r/3}(p) \setminus \gamma_i$ containing $B_{r/4}(p)$ = convex

$\Rightarrow$ repeat for each $\gamma_i \Rightarrow$ get convex $\Omega \supset B_{r/4}(p)$ bounded by geodesics $\{\gamma_i\}$



D. Claim: for any $p \in M$, $\exists\, r_p > 0$ s.t. $B_{r_p}(q)$ = normal and convex $\forall\, q \in B_{r_p}(p)$

already know $\exists\, \delta > 0$ st. $B_{2\delta}(q)$ = normal $\forall\, q \in B_\delta(p)$

now $(x,y) \in B_\delta(p) \times B_\delta(p) \longmapsto d(x,y)^2$ = smooth

and $\nabla_x^2 d(x,y)^2 > 0$ @ $x = y = p$

and $(x,y) \longmapsto \nabla_x^2 d(x,y)^2$ = smooth on $B_\delta(p) \times B_\delta(p)$

$\Rightarrow \exists\, r_p < \delta/3$ st. $\nabla_x^2 d(x,y)^2 > 0 \quad \forall\, x \in B_{3r_p}(y),\ \forall\, y \in B_{3r_p}(p)$

$\Rightarrow$ each $B_{r_p}(y)$ = convex

$\square$

since M closed (cpct + no boundary),

claim $\Rightarrow \exists\, r$ st every $B_r(p)$ = convex and uniformly normal

now simply take $\{x_i\}_i$ = maximal $\frac{r}{4}$-net on M

$\Rightarrow \{B_{r/4}(x_i)\}_i$ cover M

and $\{B_{r/8}(x_i)\}_i$ disjoint

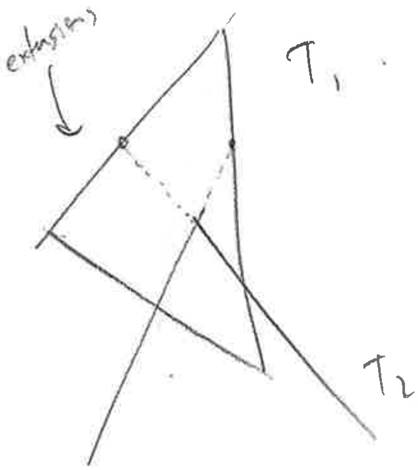
E. construct Ω_i as in part D.

subdivide each Ω_i into convex, geodesic triangles, each being contained in a convex, uniformly normal neighborhood

by part B., can just connect any vertex in $\partial\Omega$ with all other vertices in $\partial\Omega$

$\Rightarrow$ get cover of M by convex, geodesic triangles

take T_1, if any other T_i meets T_1, extend edges of T_i so they become geodesics bisecting T_1

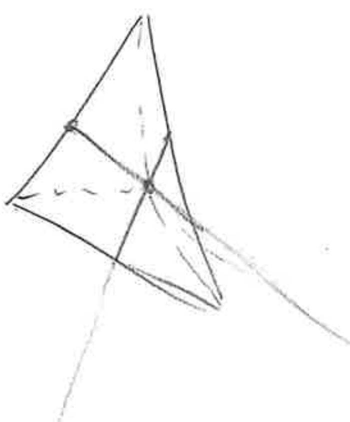


$\Rightarrow$ get collection of geodesics bisecting T_1

$\Rightarrow$ subdivides T_1 into disjoint, convex, geodesic polygons

$\Rightarrow$ subdivide each polygon into triangles

now take T_2, and as before if any other T_i meets T_2, extend edges so they bisect T_2



$\Rightarrow$ get collection of geodesics in T_2 which either bisect T_2 and/or bisect some triangle in subdivision of T_1

$\Rightarrow$ obtain collection of disjoint convex, geodesic polygons subdividing T_2

$\Rightarrow$ subdivide into triangles

$\Rightarrow$ may modify T_1 but will only subdivide triangles in T_1 further

$\Rightarrow$ repeat for $T_3, T_4, T_5, \ldots$ etc.