Math 60670 Homework 7

Due Monday, April 1.

Q1. A Riemannian submanifold $M \subset (\bar{M}, \bar{g})$ (with the induced metric $g = \bar{g}|_{TM}$), is called totally geodesic if for every $p \in M$, $v \in T_pM$, the $\bar{g}$-geodesic γ with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$ lies in M. Show the following are equivalent:

A. M is totally geodesic,
B. Every g-geodesic in M is also a $\bar{g}$-geodesic in $\bar{M}$,
C. The second fundamental form of M vanishes.

Q2. A geodesic triangle in a Riemannian 2-manifold (M^2, g) is a domain Ω with piecewise-smooth boundary $\partial\Omega$ consisting of three geodesics meeting at three vertices. If M has constant Gauss curvature K, show that the sum of interior angles of any geodesic triangle is $\pi + KA$, where A is the area of Ω.

Q3. Let ω be a 1-form on (M, g). Show that

$$(R(X,Y)\omega)(Z) := (\nabla^2_{X,Y}\omega)(Z) - (\nabla^2_{Y,X}\omega)(Z) = -\omega(R(X,Y)Z).$$

Q4. Show the scalar curvature Scal at a point $p \in M$ can be written

$$\mathrm{Scal}(p) = \frac{n}{|S^{n-1}|}\int_{S^{n-1}} \mathrm{Ric}_p(\theta,\theta)d\theta,$$

where S^{n-1} is the Euclidean $(n-1)$-sphere, and $|S^{n-1}|$ is its $(n-1)$-volume.