Q1: let g, ∇ be induced metric, connection on M

take $p \in M$, $v \in T_pM$

let $\gamma = g$-geodesic s.t. $\gamma(0) = \bar\gamma(0) = p$, $\gamma'(0) = \bar\gamma'(0) = v$

$\bar\gamma = \bar g$-geodesic

M tot. geodesic $\Rightarrow$ $\bar\gamma$ lies in M

(A)

$\Rightarrow \bar\gamma' \in T_{\bar\gamma}M$

$\Rightarrow \nabla_{\bar\gamma'}\bar\gamma' = \pi(\bar\nabla_{\bar\gamma'}\bar\gamma') = 0$

$\Rightarrow \bar\gamma = \gamma$ by uniqueness of ODEs

$\Rightarrow$ every g-geodesic $=$ $\bar g$-geodesic (B)

since p, v arbitrary

spse $\gamma \equiv \bar\gamma$ $\forall$ choices of p, v (B)

$\Rightarrow \nabla_{\gamma'(0)}\gamma'(0) = \bar\nabla_{\bar\gamma'(0)}\bar\gamma'(0)$

$\Rightarrow B(v,v)|_p = \bar\nabla_{\bar\gamma'(0)}\bar\gamma'(0) - \nabla_{\bar\gamma'(0)}\gamma'(0) = 0$ (C)

$\Rightarrow B = 0 \ \forall p, v$

spse $B \equiv 0$ $\Rightarrow \bar\nabla_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + B(\gamma', \gamma')$

(C)

$= 0$

$\Rightarrow \gamma = \bar\gamma \Rightarrow \bar\gamma$ lies in $M \Rightarrow M$ tot. geodesic

(A)

Q : recall Gauss - Bonnet says : if Ω curved polygon in (M, g)

$$\Rightarrow \int_\Omega K + \int_{\partial\Omega} k + \sum \varepsilon_i = 2\pi \chi(\Omega)$$

Gauss curvature

geodesic curvature of ∂

exterior angles

Euler characteristic

if Ω geodesic triangle $\Rightarrow \chi = V - E + F$

$= 3 - 3 + 1$

$= 1$

and $\partial\Omega$ has zero curvature

and $\underbrace{\sum \varepsilon_i}_{\text{ext. angles}} = 3\pi - \underbrace{\sum \theta_i}_{\text{interior angles}}$

since $K = \text{const}$

$$\Rightarrow K|\Omega| + 3\pi - \sum \theta_i = 2\pi$$

$$\Rightarrow \sum \theta_i = \pi + K|\Omega|$$

$\underline{a}$ choose normal coords @ p $\Rightarrow$ $\nabla \partial_i |_p = 0$

$$\Rightarrow (\nabla^2_{\partial_i, \partial_j} \omega)(\partial_k) = \partial_i \left[(\nabla_{\partial_j} \omega)(\partial_k) \right] - (\nabla_{\cancelto{0}{\nabla_{\partial_i}\partial_j}} \omega)(\partial_k)$$

$$- (\nabla_{\partial_j} \omega)(\cancelto{0}{\nabla_{\partial_i} \partial_k})$$

$$= \partial_i \left[(\nabla_{\partial_j} \omega)(\partial_k) \right] \quad @ \ p$$

$\omega = \omega_i dx^i$

$$= \partial_i (\partial_j \omega_k - \omega(\nabla_j \partial_k))$$

$$= \partial_i \partial_j \omega_k - (\nabla_i \omega)(\cancelto{0}{\nabla_j \partial_k}) - \omega(\nabla_i \nabla_j \partial_k)$$

$$= \partial_i \partial_j \omega_k - \omega(\nabla_i \nabla_j \partial_k) \quad @ \ p$$

$$\Rightarrow (\nabla^2_{i,j} \omega - \nabla^2_{j,i} \omega)(\partial_k) = \cancel{\partial_i \partial_j \omega_k} - \omega(\nabla_i \nabla_j \partial_k)$$

$$- \cancel{\partial_j \partial_i \omega_k} + \omega(\nabla_j \nabla_i \partial_k)$$

$$= -\omega(\nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k)$$

$$= -\omega(R(\partial_i, \partial_j) \partial_k) \quad @ \ p$$

since both sides tensorial

$$\Rightarrow (\nabla^2_{X,Y} \omega - \nabla^2_{Y,X} \omega)(Z) = X^i Y^j Z^k (\nabla^2_{i,j} \omega - \nabla^2_{j,i} \omega)(\partial_k)$$

$$= X^i Y^j Z^k (-\omega(R(\partial_i, \partial_j)\partial_k))$$

$$= -\omega(R(X,Y)Z)$$

Q · let Q = quadratic bilinear form

↳ $\displaystyle\int_{S^{n-1}} Q(\theta,\theta)\, d\theta = \sum_{i,j} Q(e_i, e_j) \int_{S^{n-1}} (\theta\cdot e_i)(\theta\cdot e_j)\, d\theta$

where $e_1 \dots e_n$ = std basis of $\mathbb{R}^n$

if $i \neq j$, choose $R \in O(n)$ st. $R(e_i) = e_i$
$R(e_j) = -e_j$

↳ R isometry of S^{n-1}

and $\displaystyle\int_{S^{n-1}} (\theta\cdot e_i)(\theta\cdot e_j)\, d\theta = -\int_{R(S^{n-1})} (R(\theta)\cdot e_i)(R(\theta)\cdot e_j)\, d\theta$

$\displaystyle = -\int_{S^{n-1}} (\theta\cdot e_i)(\theta\cdot e_j)\, d\theta = 0$

if $i = j$, choose $R \in O(n)$ st $Re_i = e_j$

$\displaystyle\Rightarrow \int_{S^{n-1}} (\theta\cdot e_i)^2\, d\theta = \int_{R(S^{n-1})} (R(\theta)\cdot e_j)^2\, d\theta = \int_{S^{n-1}} (\theta\cdot e_j)^2\, d\theta$

$\displaystyle\Rightarrow n \int_{S^{n-1}} (\theta\cdot e_i)^2\, d\theta = \sum_{i=1}^{n} \int_{S^{n-1}} (\theta\cdot e_i)^2\, d\theta$

$\displaystyle = \int_{S^{n-1}} |\theta|^2\, d\theta = |S^{n-1}|$

So $$\int_{S^{n-1}} Q(\theta,\theta)\,d\theta = \sum_{i,j} Q(e_i,e_j) \int_{S^{n-1}} (\theta\cdot e_i)(\theta\cdot e_j)\,d\theta$$

$$= \sum_i Q(e_i,e_i)\,\frac{|S^{n-1}|}{n}$$

$$= \frac{|S^{n-1}|}{n}\,\mathrm{tr}\,Q$$