Math 60670 Homework 6

Due Wednesday, March 25.

Q1. Prove the second Bianchi identity

$$(\nabla_T R)(X, Y, Z, W) + (\nabla_X R)(Y, T, Z, W) + (\nabla_Y R)(T, X, Z, W) = 0.$$

Hint: use normal coordinates.

Q2. If $M^n \subset (\bar{M}^{n+k}, \bar{g})$ be an embedded (or immersed) submanifold, and B for the second fundamental form, the mean curvature H of M is defined to be the trace of the second fundamental form:

$$H := tr_g B \equiv g^{ij} B_{ij} \equiv \sum_i B(e_i, e_i),$$

where $g = \bar{g}|_M$ is the induced metric on M and $\{e_i\}_i$ is any orthonormal basis for T_pM. We say M is minimal if $H = 0$.

A. The catenoid in $\mathbb{R}^3$ is the surface obtained by revolving the curve $r = \cosh(z)$ around the z-axis. Prove the catenoid is minimal in $\mathbb{R}^3$.

B. The Clifford torus is a torus embedded in $S^3 \subset \mathbb{R}^4$ defined the relation

$$\Sigma := \{(x, y) \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap S^3 : |x| = |y|\}.$$

Prove Σ is minimal in S^3. Hint: Σ can be parameterized by the map $F(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi)$.

Q3. **A.** Let (M^2, g) be a 2-dimensional Riemannian manifold, and $u \in C^\infty(M)$. Define the conformally-changed metric $\tilde{g} = e^{2u} g$. Write ∇, $\tilde{\nabla}$ for the Levi-Civita connections w.r.t. g, $\tilde{g}$, respectively. If (x^1, x^2) are coordinates for M, show that

$$\tilde{\nabla}_i \partial_j = (\partial_i u)\partial_j + (\partial_j u)\partial_i - g_{ij} g^{kp} (\partial_k u)\partial_p + \nabla_i \partial_j.$$

and deduce that

$$\tilde{\nabla}_X Y = X(u)Y + Y(u)X - g(X, Y)\mathrm{grad}_g(u) + \nabla_X Y$$

where $\mathrm{grad}_g(u)$ is the gradient vector of u w.r.t. g.

B. Use part A to prove the relation

$$\tilde{K} = e^{-2u}(-\Delta_g u + K)$$

where $K, \tilde{K}$ are the Gauss curvatures of $g, \tilde{g}$, and $\Delta_g u \equiv \mathrm{tr}_g \nabla^2 u$ is the g-connection Laplacian of u. Hint: use geodesic normal coordinates for (M, g).