Q : choose normal coords x^i @ $o \in M$

$$\Rightarrow \nabla \partial_i |_o = 0$$

then: $\nabla_p R_{ijkl} = \partial_p R_{ijkl}$ @ o

$$= \langle \nabla_p \nabla_i \nabla_j \partial_k - \nabla_p \nabla_j \nabla_i \partial_k , \partial_l \rangle \quad @ \; o$$

$$\Rightarrow \nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl}$$

$$= \langle \nabla_p \nabla_i \nabla_j \partial_k - \nabla_p \nabla_j \nabla_i \partial_k + \nabla_i \nabla_j \nabla_p \partial_k - \nabla_i \nabla_p \nabla_j \partial_k + \nabla_j \nabla_p \nabla_i \partial_k - \nabla_j \nabla_i \nabla_p \partial_k , \partial_l \rangle$$

$$= R(\partial_p, \partial_i, \underbrace{\nabla_j \partial_k}_{0}, \partial_l) + R(\partial_j, \partial_p, \underbrace{\nabla_i \partial_k}_{0}, \partial_l) + R(\partial_i, \partial_j, \underbrace{\nabla_p \partial_k}_{0}, \partial_l)$$

$$= 0$$

by tensoriality, $(\nabla_T R)(X, Y, Z, W) + (\nabla_X R)(Y, T, Z, W) + (\nabla_Y R)(T, X, Z, W) = 0$ @ o

Q2.A param catenoid as

$$F(\theta, z) = (\cosh(z)\cos\theta,\ \cosh(z)\sin\theta,\ z)$$

$\hookrightarrow$ $\partial_\theta F = (-\cosh(z)\sin\theta,\ \cosh(z)\cos\theta,\ 0)$

$\partial_z F = (\sinh(z)\cos\theta,\ \sinh(z)\sin\theta,\ 1)$

$\Rightarrow$ $g_{zz} = \sinh^2(z) + 1$, $g_{z\theta} = 0$, $g_{\theta\theta} = \cosh^2(z)$

$= \cosh^2(z)$

a choice of unit normal is $\nu = \dfrac{(\cos\theta,\ \sin\theta,\ -\sinh(z))}{\cosh(z)}$

$\hookrightarrow$ $\partial_{\theta\theta} F = (-\cosh(z)\cos\theta,\ -\cosh(z)\sin\theta,\ 0)$

$\partial_{\theta z} F = (-\sinh(z)\sin\theta,\ \sinh(z)\cos\theta,\ 0)$

$\partial_{zz} F = (\cosh(z)\cos\theta,\ \cosh(z)\sin\theta,\ 0)$

$\Rightarrow$ $h_{zz} = \partial^2_{zz} F \cdot \nu$, $h_{z\theta} = 0$, $h_{\theta\theta} = -1$

$= 1$

So scalar mean curvature $H = g^{ij} h_{ij}$

$= \dfrac{1}{\cosh^2}(h_{zz} + h_{\theta\theta})$

$= 0$ ✓

Q2.B param Clifford torus as

$$F(\theta, \varphi) = \frac{1}{\sqrt{2}} (e^{i\theta}, e^{i\varphi}) \subset S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$$

$\hookrightarrow$ $\partial_\theta F = \frac{1}{\sqrt{2}} (i e^{i\theta}, 0)$

$\partial_\varphi F = \frac{1}{\sqrt{2}} (0, i e^{i\varphi})$

$\Rightarrow$ $g_{\theta\theta} = \frac{1}{2}$, $g_{\theta\varphi} = 0$, $g_{\varphi\varphi} = \frac{1}{2}$

a choice of unit normal is $\nu = \frac{1}{\sqrt{2}} (e^{i\theta}, -e^{i\varphi})$

(being $\perp$ F, $\partial_\theta F$, $\partial_\varphi F$)

$\hookrightarrow$ $\partial_{\theta\theta} F = -\frac{1}{\sqrt{2}} (e^{i\theta}, 0)$

$\partial_{\theta\varphi} F = 0$

$\partial_{\varphi\varphi} F = -\frac{1}{\sqrt{2}} (0, e^{i\varphi})$

$\Rightarrow$ $h_{\theta\theta} = \partial^2_{\theta\theta} F \cdot \nu$, $h_{\theta\varphi} = 0$, $h_{\varphi\varphi} = \frac{1}{2}$

$= -\frac{1}{2}$

$\Rightarrow$ $H = g^{ij} h_{ij} = 2(h_{\theta\theta} + h_{\varphi\varphi})$

$= 0$

Q3A. we have

$$\widetilde{\Gamma}_{ij}^{k} = \frac{1}{2}\tilde{g}^{kp}\left(\partial_i \tilde{g}_{jp} + \partial_j \tilde{g}_{ip} - \partial_p \tilde{g}_{ij}\right)$$

$$= \frac{1}{2} e^{-2u} g^{kp}\left(\partial_i (e^{2u} g_{jp}) + \partial_j (e^{2u} g_{ip}) - \partial_p (e^{2u} g_{ij})\right)$$

$$= (\partial_i u)\delta_{jk} + (\partial_j u)\delta_{ik} - g^{kp}(\partial_p u)\, g_{ij} + \Gamma_{ij}^{k}$$

$$\Rightarrow \widetilde{\nabla}_{\partial_i}\partial_j = \widetilde{\Gamma}_{ij}^{k}\,\partial_k$$

$$= (\partial_i u)\partial_j + (\partial_j u)\partial_i - g_{ij} g^{kp}(\partial_p u)\partial_k + \nabla_{\partial_i}\partial_j$$

$$\Rightarrow \widetilde{\nabla}_X Y = X(Y^j)\partial_j + X^i Y^j \widetilde{\nabla}_{\partial_i}\partial_j$$

$$= X(Y^j)\partial_j + X^i Y^j\left((\partial_i u)\partial_j + (\partial_j u)\partial_i - g_{ij} g^{kp}\partial_p u\, \partial_k + \nabla_{\partial_i}\partial_j\right)$$

$$= \nabla_X Y + X(u)Y + Y(u)X - g(X,Y)\,\mathrm{grad}(u)$$

(recalling that $\mathrm{grad}(u) = g^{kp}(\partial_p u)\partial_k$)

notice that if $Y|_p = 0$ then $\widetilde{\nabla}_X Y = \nabla_X Y$

Q3.B. choose normal coords for (M, g) @ p

so $g_{ij} = \delta_{ij}$ @ p, $g^{ij} = \delta_{ij}$ @ p

$\partial_k g_{ij} = \partial_k g^{ij} \rightsquigarrow 0$ @ p

$\Gamma_{ij}^k = 0$ @ p

$\Delta f = (\partial_1^2 + \partial_2^2) f$ @ p

for shorthand write $u_i = \partial_i u$ and $u_{ij} = \partial^2_{ij} u$

look @ $\tilde{R}_{1221} = \tilde{g}(\tilde{\nabla}_1 \tilde{\nabla}_2 \partial_2 - \tilde{\nabla}_2 \tilde{\nabla}_1 \partial_2, \partial_1)$

$$= e^{2u} g(\tilde{\nabla}_1(\tilde{\Gamma}_{22}^p \partial_p) - \tilde{\nabla}_2(\tilde{\Gamma}_{12}^p \partial_p), \partial_1)$$

$$= e^{2u}(\partial_1 \tilde{\Gamma}_{22}^1 + \tilde{\Gamma}_{22}^p \tilde{\Gamma}_{1p}^1 - \partial_2 \tilde{\Gamma}_{12}^1 - \tilde{\Gamma}_{12}^p \tilde{\Gamma}_{2p}^1)$$

compute: $\partial_1 \tilde{\Gamma}_{22}^1 = \partial_1(\cancel{2u_2\delta_{12}} - g_{22} g^{1p} u_p + \Gamma_{22}^1)$

$$= -\delta_{22}\delta_{1p} u_{1p} + \partial_1 \Gamma_{22}^1 \quad @\ p$$

$$= -u_{11} + \partial_1 \Gamma_{22}^1$$

$\partial_2 \tilde{\Gamma}_{12}^1 = \partial_2(u_1 \cancel{\delta_{21}} + u_2 \delta_{11} - g_{12} g^{1p} u_p + \Gamma_{12}^1)$

$$= u_{22} - \delta_{12}\delta_{1p} u_{2p} + \partial_2 \Gamma_{12}^1 \quad @\ p$$

$$= u_{22} + \partial_2 \Gamma_{12}^1 \quad @\ p$$

$$\tilde{\Gamma}_{22}^{p}\tilde{\Gamma}_{1p}^{1} = (2u_2\delta_{2p} - u_p)(u_1\delta_{p1} + u_p - \delta_{1p}u_1) \quad @\, p$$

$$= 2u_2^2 - u_1^2 - (u_1^2 + u_2^2) + u_1^2$$

$$= u_2^2 - u_1^2$$

$$\tilde{\Gamma}_{12}^{p}\tilde{\Gamma}_{2p}^{1} = (u_1\delta_{2p} + u_2\delta_{1p})(u_2\delta_{1p} - \delta_{2p}u_1) \quad @\, p$$

$$= -u_1^2 + u_2^2$$

$$\underline{\text{So}}: \quad \tilde{R}_{1221} = e^{2u}\left(-u_{11} + \partial_1\Gamma_{22}^{1} - u_{22} - \partial_2\Gamma_{12}^{1}\right) \quad @\, p$$

$$= e^{2u}(-\Delta u + R_{1221})$$

$$= e^{2u}(-\Delta u + K)$$

$$\Rightarrow \tilde{K} = \frac{\tilde{R}_{1221}}{\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2} = e^{-4u}\tilde{R}_{1221}$$

$$= e^{-2u}(-\Delta u + K)$$