Math 60670 Homework 5

Due March 4.

Problem 1. Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be a diffeomorphism between connected Riemannian manifolds M, $\tilde{M}$.

A. Suppose ϕ is a Riemannian isometry. Show that ϕ takes geodesics to geodesics, and deduce that $\phi \circ \exp_p = \exp_{\phi(p)} \circ D\phi|_p$ whereever this is defined.

B. Let $\tilde{\phi}$ be another Riemannian isometry $(M, g) \to (\tilde{M}, \tilde{g})$, and suppose there is a point $p \in M$ so that $\phi(p) = \tilde{\phi}(p)$ and $D\phi|_p = D\tilde{\phi}|_p$. Show that $\phi = \tilde{\phi}$.

C. Show that ϕ is a Riemannian isometry if and only if ϕ preserves distances, i.e. $d_g(p, q) = d_{\tilde{g}}(\phi(p), \phi(q))$ for all $p, q \in M$.

D. (Bonus) Show that part C holds even if ϕ is only assumed to be a distance-preserving *homeomorphism*.

Problem 2. Let $\mathbf{U}^2$ denote the hyperbolic plane, i.e. the upper half-plane in $\mathbb{R}^2$ with metric $h = (dx^2 + dy^2)/y^2$. Let $\mathrm{SL}(2, \mathbb{R})$ denote the group of 2×2 real matrices of determinant 1, and define an action of $A \in \mathrm{SL}(2, \mathbb{R})$ on points $z = x + iy \in \mathbf{U}^2 \subset \mathbb{C}$ by

$$A \cdot z = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

A. Show this defines a smooth action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbf{U}^2$ by isometries of the hyperbolic metric.

B. Show that geodesics in $\mathbf{U}^2$ are vertical half-lines and half-circles that intersect the "boundary" $\{y = 0\}$ orthogonally. Hint: Use part A and Problem 1A.

C. Deduce that every geodesic in $\mathbf{U}^2$ can be extended or all time, i.e. $\mathbf{U}^2$ is complete.