

Math 60670 Homework 4

Due February 25.

Problem 1: A. The Laplacian of a function f is defined to be the trace of the Hessian: $\Delta f = \text{tr}_g(\nabla^2 f)$. Show that one can alternatively write

$$\Delta f = \text{div}(\text{grad} f),$$

and hence in coordinates

$$\Delta f = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f), \quad g = \det g_{ij}.$$

B. A function $u \in C^2(M)$ is called harmonic if $\Delta u = 0$. If (M, g) is compact with no boundary, prove that any harmonic function is necessarily constant. Hint: Use HW3 Problem 3.

Problem 2: A. For v in some interval I , let $(r(v), z(v))$ be a smooth, regular curve in the $r - z$ plane, with $r > 0$. Show that

$$F(\theta, v) = (r(v) \cos(\theta), r(v) \sin(\theta), z(v)), \quad t \in I, \theta \in \mathbb{R}.$$

is a well-defined immersion. The image of F is the surface of revolution obtained by rotating the curve $(r(v), z(v))$ about the z -axis. The lines corresponding to $\theta = \text{const}$, $v = \text{const}$ are called the meridians, parallels (respectively).

B. Show that the induced metric $F^* g_{\text{eucl}}$ in (θ, v) coordinates is given by

$$g_{11} = r^2, \quad g_{12} = 0, \quad g_{22} = r'^2 + z'^2.$$

C. Show that that $\gamma = (\theta(t), v(t))$ is a geodesic if and only if

$$\ddot{\theta} + \frac{2rr'}{r^2} \dot{\theta} \dot{v} = 0 \tag{1}$$

$$\ddot{v} - \frac{rr'}{r'^2 + z'^2} \dot{\theta}^2 + \frac{r'r'' + z'z''}{r'^2 + z'^2} \dot{v}^2 = 0. \tag{2}$$

D. Deduce meridians are always geodesics. When is a parallel geodesic?

E. Show that equations (1), (2) have the following “first order” interpretation: (2) is (except for meridians, parallels) equivalent to the fact that the “energy” $|\gamma'(t)|^2$ is constant; (1) is equivalent to the “Clairaut’s relation:”

$$r(v(t)) \cos \beta(t) = \text{const},$$

where $\beta(t)$ is the angle made between $\gamma'(t)$ and the parallel intersecting $\gamma(t)$.

F. Use Clairaut’s relation to show that a geodesic of the paraboloid $r(t) = t, z(t) = t^2$ which is not a parallel or meridian must intersect itself infinitely-many times.