Math 60670 Homework 3

Due Wednesday, February 18.

Problem 1: Let ∇ be a torsion-free linear connection, and ω a 1-form. Show that

$$d\omega(X,Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X)$$

for any $X, Y \in \mathcal{X}(M)$.

Problem 2: Let ∇ be a linear connection. Given a curve $\gamma(t) : I \to M$ and a vector $V \in T_{\gamma(s)}M$ (for some $s \in I$), let us write $P_{\gamma,V}(t)$ for the parallel transport of V along γ with respect to ∇. If $X, Y \in \mathcal{X}(M)$ and $p \in M$, show that

$$\nabla_X Y|_p = \lim_{t\to 0} \frac{1}{t}(P_{\gamma,Y(\gamma(t))}(0) - Y(0))$$

for any curve $\gamma : (-\epsilon, \epsilon) \to M$ satisfying $\gamma(0) = p$, $\gamma'(0) = X$.

Problem 3: Let (M, g) be an oriented Riemannian manifold. If ω is a k-form on M, and X is a vector field, then $\iota_X \omega$ is the $(k-1)$-form obtained by contracting against X in the first slot:

$$(\iota_X \omega)(V_1, \ldots, V_{k-1}) := \omega(X, V_1, \ldots, V_{k-1}).$$

One way to define the divergence operator div : $\mathcal{X}(M) \to C^\infty(M)$ is is by the relation

$$d(\iota_X dV) = \mathrm{div}(X) dV.$$

A. Show that if M is a compact manifold-with-boundary, then

$$\int_M \mathrm{div}(X) dV = \int_{\partial M} < X, N > dV_{\partial M},$$

where $dV_{\partial M}$ is the volume-form on ∂M with the inducted metric and orientation, and N is the outwards pointing conormal of ∂M.

B. Show that if $\phi \in C^\infty(M)$, then div satisifes the following product rule:

$$\mathrm{div}(\phi X) =< \mathrm{grad}\phi, X > +\phi \mathrm{div}(X).$$

C. Using the above or otherwise, show that in local coordinates div can be written as

$$\operatorname{div}(X) = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}X^i), \quad g = \det(g_{ij}).$$

D. Show that if ∇ is the Levi-Civita connection, then

$$\operatorname{div}(X) = \operatorname{tr}(\nabla X) \equiv (\nabla X)^i_i \equiv \sum_i < e_i, \nabla_{e_i} X >,$$

(e_i being any an ON basis of the tangent space).