# Math 60670 Homework 2

Due February 11.

**Problem 1:** Let $\phi : M \to \bar{M}$ be a smooth map, $X, Y$ vector fields on $M$, $\bar{X}, \bar{Y}$ vector fields on $\bar{M}$, and suppose $\bar{X}|_{\phi(p)} = D\phi(X|_p)$, $\bar{Y}|_{\phi(p)} = D\phi(Y|_p)$ for every $p \in M$. Show that $[\bar{X}, \bar{Y}]|_{\phi(p)} = D\phi([X, Y]|_p)$.

**Problem 2:** Show there are vector fields $X_1, X_2, Y$ on $\mathbb{R}^2$, such that on the $x^1$-axis $X_1 = X_2 = (1, 0)$ and $Y = (0, 1)$, but such that the Lie derivatives $L_{X_1}Y \neq L_{X_2}Y$. (So, one cannot use the Lie derivative to define a reasonable notion of "derivative of vector field along a curve").

**Problem 3:** Let $(x^i)$, $(y^\alpha)$ be local coordinates defined in some $U \subset M$. Suppose $A$ is a $(1, 2)$-tensor field which in the $x$-coordinate system can be expressed as

$$A = A^i_{jk}(x)\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k.$$

Show that in the $y$-coordinate system the components of $A$ are

$$A^a_{bc}(y = y(x)) = \frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\frac{\partial x^k}{\partial y^c}A^i_{jk}(x).$$

Use this to show explicitly that the result of contracting the $i$, $j$ indices together is independent of choice of coordinates.

**Problem 4:** Show that $T$ is a smooth $(k, l)$-tensor field on $M$ if and only if $T$ is a smooth, $\mathbb{R}$-multilinear function from $k$ 1-forms and $l$ vector fields to $\mathbb{R}$, which is also multilinear over $C^\infty(M)$. By "smooth" we mean that if $X_1, \ldots, X_l \in \mathcal{X}(M)$, $\omega_1, \ldots, \omega_k \in \mathcal{X}^*(M)$, then $T(\omega_1, \ldots, \omega_l, X_1, \ldots, X_k) \in C^\infty(M)$.

**Problem 5:** **A.** Let $(M, g)$ be an oriented Riemannian $n$-manifold, and let $(x^1, \ldots, x^n) : U \subset \mathbb{R}^n \to M$ be coordinates compatible with the orientation in the sense that $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ is a positively-oriented basis. Show that the volume form $dV = \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$, and deduce that if $f : M \to \mathbb{R}$

is supported in the $x$-coordinate chart, then

$$\int_M f dV = \int_{U\subset\mathbb{R}^n} f(x^1,\ldots,x^n)\sqrt{\det g_{ij}}\, dx^1\cdots dx^n.$$

**B.** Let $(y^1,\ldots,y^n)$ be coordinates on $M$, which may or may not be positively oriented, and which overlap with the $x$-coordinate chart. If $g_{\alpha\beta}$ is the metric in $y$-coordinates, show that $\det(g_{ij}) = \det(\frac{\partial y^p}{\partial x^q})^2 \det(g_{\alpha\beta})$.

**C.** Prove that $\int_M f dV$ is independent of choice of orientation on $M$.