Q1 If $\gamma(t)$ = curve in M with $\gamma(0) = p$, $\gamma'(0) = X$

then $\frac{d}{dt}\Big|_{t=0} (f \circ \varphi)(\gamma(t)) = X(f \circ \varphi)|_p$

$$\frac{d}{dt}\Big|_{t=0} f(\varphi(\gamma(t))) = (D\varphi|_p X)(f)|_{\varphi(p)} \quad \text{since} \quad \frac{d}{dt}\Big|_{t=0} \varphi(\gamma(t)) = D\varphi|_p X$$

$$= \bar{X}(f)|_{\varphi(p)}$$

and similarly $Y(f \circ \varphi)|_p = \bar{Y}(f)|_{\varphi(p)}$

so $\bar{X}(\bar{Y}(f))|_q = X(\bar{Y}(f) \circ \varphi)|_{\varphi^{-1}(q)}$

$$= X(Y(f \circ \varphi))|_{\varphi^{-1}(q)}$$

$$\Rightarrow [\bar{X}, \bar{Y}]f|_q = \bar{X}(\bar{Y}f)|_q - \bar{Y}(\bar{X}(f))|_q$$

$$= X(Y(f \circ \varphi))|_{\varphi^{-1}(q)} - Y(X(f \circ \varphi))|_{\varphi^{-1}(q)}$$

$$= [X, Y](f \circ \varphi)|_{\varphi^{-1}(q)}$$

$$= (D\varphi|_{\varphi^{-1}(q)} [X, Y])(f)|_q$$

Q: let $X_1 = (1, 0)$

$X_2 = (1, x^2)$ $\Rightarrow X_1 = X_2 = (1,0)$ on x^1-axis

$Y = (0, 1)$

but $L_{X_1} Y = (0,0)$ while $L_{X_2} Y = -\partial_2(x^2)\,\partial_2 = (0,-1)$

Q3 : by chain rule, $dx^i = \frac{\partial x^i}{\partial y^a} dx^a$

$$\frac{\partial}{\partial x^k} = \frac{\partial y^c}{\partial x^k} \frac{\partial}{\partial y^c}$$

$$\Rightarrow A^k_{ij} \, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

$$= \underbrace{A^k_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial y^c}{\partial x^k}}_{= A^c_{ab}} dy^a \otimes dy^b \otimes \frac{\partial}{\partial y^c}$$

$$(\mathrm{tr}\, A)_b = A^a_{ab} = A^k_{ij} \underbrace{\frac{\partial x^i}{\partial y^a} \frac{\partial y^a}{\partial x^k}}_{\substack{\text{by chain rule} \\ \frac{\partial x^i}{\partial x^k} = \delta_{ik}}} \frac{\partial x^j}{\partial y^b} = A^i_{ij} \frac{\partial x^j}{\partial y^b}$$

$$= (\mathrm{tr}\, A)_j \frac{\partial x^j}{\partial y^b}$$

Q : let's show $T \in \mathcal{T}^{(1,1)}(M) \iff T : \mathfrak{X}(M) \otimes \mathfrak{X}^*(M) \longrightarrow C^\infty(M)$ smooth, linear over $C^\infty(M)$

general case is verbatim

$(\Rightarrow)$ trivial since in coords $T = T_i^{\,j}\, dx^i \otimes \partial_j$

$\Rightarrow T(fX, g\omega) = f X^i g \omega_j T_i^{\,j}$

$= f g T(X, \omega)$

for $X \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$
$\omega \in \mathfrak{X}^*(M)$

$(\Leftarrow)$ in coords we have $T(X^i \partial_i, \omega_j dx^j)$

$= X^i \omega_j T(\partial_i, dx^j)$

$= \underbrace{\left(T(\partial_i, dx^j)\, dx^i \otimes \partial_j\right)}_{= (1,1)\text{-tensor}}(X, \omega)$

Q : recall general property about n-forms in $\mathbb{R}^n$

if $e_1 \dots e_n$ = std basis, $\omega^1, \dots, \omega^n$ = dual basis

$$\Rightarrow (\omega^1 \wedge \dots \wedge \omega^n)(v_1, \dots, v_n) = \det \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

let E_i = oriented, g-ON basis of T_pM

θ^i = dual basis

$$\Rightarrow dV\big|_p = \theta^1 \wedge \dots \wedge \theta^n$$

if x^i coords near p $\Rightarrow \partial_i = a_i^j E_j$

$$\Rightarrow dV(\partial_1, \dots, \partial_n) = \det \begin{bmatrix} a_1^j & a_2^j & \cdots & a_n^j \end{bmatrix}$$

$$= \det A \quad \text{for } A_{ij} := a_j^i$$

now $g_{ij} = g(\partial_i, \partial_j) = a_i^p a_j^q g(e_p, e_q)$

$$= a_i^p a_j^q \delta_{pq}$$

$$= a_i^p a_j^p = (A^T A)_{ij}$$

$$\Rightarrow \det(g_{ij}) = (\det A)^2$$

$$\Rightarrow \det A = \sqrt{\det(g_{ij})}$$ since ∂_i positively-oriented (so $\det A > 0$)

$$\Rightarrow dV = \det A \, dx^1 \wedge \dots \wedge dx^n = \sqrt{\det(g_{ij})} \, dx^1 \wedge \dots \wedge dx^n$$

B. we have $\frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i}$

so $g_{\alpha\beta} = g\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right) = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

$$= \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} g_{ij}$$

$\Rightarrow \det(g_{\alpha\beta}) = \det\left(\frac{\partial x^i}{\partial y^\alpha}\right)^2 \det(g_{ij})$

C. first spse $f : M \to \mathbb{R}$ supported in both (x^i) and (y^α) coord charts

(here (x^i) positively oriented, (y^α) arbitrary)

$\hookrightarrow \int f\,dV = \int_{\mathbb{R}^n} f(x^1,\dots,x^n)\sqrt{\det g_{ij}}\,dx^1\cdots dx^n$

$$= \int_{\mathbb{R}^n} f(y^1,\dots,y^n)\left|\det\left(\frac{\partial x^i}{\partial y^\alpha}\right)\right|\sqrt{\det g_{ij}}\,dy^1\cdots dy^n$$

(by area formula)

$$= \int_{\mathbb{R}^n} f(y^1,\dots,y^n)\sqrt{\det g_{\alpha\beta}}\,dy^1\cdots dy^n$$

since (y^α) arbitrary, we deduce $\int f\,dV$ ind. of choice of orientation

(eg. take $y^1 = -x^1, y^2 = x^2, \dots y^n = x^n$ to have opposite orientation)

for general $f: M \to \mathbb{R}$, consider $\{\varphi_i\}$ = partition of unity on M
subordinate to some ~~oriented~~ atlas

$\hookrightarrow \int f\, dV = \sum_i \int f \varphi_i \, dV$

$\uparrow$ ind. of orientation by previous argument

$\Rightarrow \int f\, dV$ = ind. of orientation also