Math 60670 Homework 1

Due January 28.

Problem 1: (Frobienius's theorem) Let X, Y be smooth vector fields in a smooth manifold M, and suppose $[X, Y] \equiv 0$. Let $\phi_t(x)$ be the flow of X and let $\psi_s(x)$ be the flow of Y, so that

$$\phi_0(x) = \psi_0(x) = x, \quad \partial_t \phi_t(x) = X(\phi_t(x)), \quad \partial_s \psi_s(x) = Y(\psi_s(x)).$$

A. Using the definition of Lie derivative, and the group property of flows (which is that $\phi_t \circ \phi_s = \phi_{t+s}$), show that $Y(\phi_t(x)) = D\phi_t Y(x)$. Similarly, deduce that $X(\psi_s(x)) = D\psi_s X(x)$.

B. Show that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$. Hint: consider the function $F(x, s, t) = (\phi_{-t} \circ \psi_{-s} \circ \phi_t \circ \psi_s)(x)$, and show that $F(x, s, t) \equiv x$.

C. Deduce there is a smooth function $f_{t,s}(x)$ with the property that $\partial_t f_{t,s}(x) = X(f_{t,s}(x))$ and $\partial_s f_{t,s}(x) = Y(f_{t,s}(x))$.

Problem 2: Show that any smooth, connected, 1-dimensional Riemannian manifold is isometric to one of $(0, L)$, $(0, \infty)$, $\mathbb{R}$, $\mathbb{R}/L\mathbb{Z}$, for some L, where each of these manifolds inherits the metric $ds^2 = dx^2$ from $\mathbb{R}$. You may take for granted that any smooth, connected, 1-dimensional manifold is diffeomorphic to $(0, 1)$ or S^1 (though proving this is a nice exercise).

Problem 3: Using the map we defined in class or otherwise, show explicitly that the ball model and the half-space model of 2-dimensional hyperbolic space are isometric (in other words, find and verify an isometry between them).

Problem 4: Construct an isometric embedding of a flat 2-torus into $\mathbb{R}^4$. Not for credit: Do you think this torus can be isometrically embedded into $\mathbb{R}^3$? You may find the "Nash embedding theorem" interesting.