Q1 a. since $[X,Y]=0 \implies \frac{d}{dt}\Big|_{t=0} D\varphi_{-t}(Y|_{\varphi_t(p)}) = 0 \quad \forall p, t$

noting that $\varphi_{t+s}(p) = \varphi_t(\varphi_s(p))$

we compute $\frac{d}{dt}\Big|_t D\varphi_{-t}(Y|_{\varphi_t(p)})$

$$= \frac{d}{ds}\Big|_{s=0} D\varphi_{-t-s}(Y|_{\varphi_{t+s}(p)})$$

$$= D\varphi_{-t} \frac{d}{ds}\Big|_{s=0} D\varphi_{-s}(Y|_{\varphi_s(\varphi_t(p))})$$

$$= 0$$

So $D\varphi_{-t}(Y|_{\varphi_t(p)}) = \text{constant in } t = D\varphi_{-0}(Y|_{\varphi_0(p)}) = Y$

$\implies Y|_{\varphi_t(p)} = D\varphi_t(Y|_p)$

by same reasoning, we have $Y|_{\psi_s(p)} = D\psi_s(Y|_p)$ $\left(\begin{matrix}[Y,Y]=0\\ \text{trivially}\end{matrix}\right)$

$X|_{\varphi_t(p)} = D\varphi_t(X|_p)$

$X|_{\psi_s(p)} = D\psi_s(X|_p)$

B. if $F(s,t) = (\varphi_{-t} \circ \psi_{-s} \circ \varphi_t \circ \psi_s)(x)$

then $\frac{\partial F}{\partial s} = D\varphi_{-t}\left(-Y\big|_{\psi_{-s}(\varphi_t(\psi_s(p)))} + D\psi_{-s} \circ D\varphi_t\, Y\big|_{\psi_s(p)}\right)$

$= D\varphi_{-t}\left(-Y\big|_{\psi_{-s}(\varphi_t(\psi_s(p)))} + D\psi_{-s}\left(Y\big|_{\varphi_t(\psi_s(p))}\right)\right)$

$= D\varphi_{-t}\left(-Y\big|_{\psi_{-s}(\varphi_t(\psi_s(p)))} + Y\big|_{\psi_{-s}(\varphi_t(\psi_s(p)))}\right)$

$= 0$

similar argument shows $\frac{\partial F}{\partial t} = 0 \Rightarrow F = \text{const}$

$\Rightarrow \varphi_t(\psi_s(x)) = \psi_s(\varphi_t(x))$

C. set $f_{s,t}(x) = \varphi_t(\psi_s(x)) \equiv \psi_s(\varphi_t(x))$

then $\frac{\partial f}{\partial t} = X\big|_{\varphi_t(\psi_s(x))} = X\big|_f$

$\frac{\partial f}{\partial s} = Y\big|_{\psi_s(\varphi_t(x))} = Y\big|_f$

Q2. define $F(x_1, \dots, x_n) = \dfrac{(2x_1, \dots, 2x_{n-1}, 1-|x|^2)}{\underbrace{x_1^2 + \dots + x_{n-1}^2 + (1-x_n)^2}_{=: D}}$

Want to show F = isometry $\left(B_1, g_{ball} = \dfrac{4\,dx^2}{(1-|x|^2)^2}\right)$

$\longrightarrow \left(\{x_n > 0\}, g_{upper} = \dfrac{dx^2}{x_n^2}\right)$

clearly F takes $B_1 \to \{x_n > 0\}$ and is a smooth map

claim: F = diffeo $B_1 \to \{x_n > 0\}$

define the (smooth) map $G(y_1, \dots, y_n) = \dfrac{(2y_1, \dots, 2y_{n-1}, |y|^2 - 1)}{y_1^2 + \dots + y_{n-1}^2 + (1+y_n)^2}$

ETS: $G \circ F = id_{B_1}$

$\hookrightarrow$ ETS: $\dfrac{(2F_1, \dots, 2F_{n-1}, |F|^2 - 1)}{|F|^2 - F_n^2 + (1+F_n)^2} = (x_1, \dots, x_n)$

now $|F|^2 - F_n^2 + (1+F_n)^2 = \dfrac{4(|x|^2 - x_n^2)}{D^2} + \dfrac{4(1-x_n)^2}{D^2} = \dfrac{4}{D}$

and $|F|^2 - 1 = \dfrac{4(|x|^2 - x_n^2) + (1-|x|^2)^2 - D^2}{D^2}$

$= \dfrac{4}{D^2}\left[|x|^2 - x_n^2 - |x|^2 + |x|^2 x_n + x_n - x_n^2\right]$

$= \dfrac{4x_n}{D}$

so $G \circ F = \dfrac{(2F_1, \dots, 2F_{n-1}, |F|^2 - 1)}{|F|^2 - F_n^2 + (1+F_n)^2} = \dfrac{\left(\frac{4x_1}{D}, \dots, \frac{4x_{n-1}}{D}, \frac{4x_n}{D}\right)}{4/D}$

$= (x_1, \dots, x_n)$ ✓

now WTS F^* $g_{\text{upper}} = g_{\text{ball}}$

computes for $i, p < n$

$$\partial_i F_p = \frac{2\delta_{ip}}{D} - \frac{4x_p x_i}{D^2}$$

$$\partial_i F_n = -\frac{2x_i}{D} - \frac{(1-|x|^2)2x_i}{D^2} = \frac{4x_i(x_n-1)}{D^2}$$

$$\partial_n F_p = \frac{2x_p}{D^2} \cdot 2(1-x_n) = \frac{4x_p(1-x_n)}{D^2}$$

$$\partial_n F_n = -\frac{2x_n}{D} + \frac{2(1-|x|^2)(1-x_n)}{D} = \frac{2}{D^2}\left[-(|x|^2-x_n^2) + (1-x_n)^2\right]$$

so: $$\partial_i F \cdot \partial_j F = \sum_{p=1}^{n-1} \left(\frac{2\delta_{ip}}{D} - \frac{4x_p x_i}{D^2}\right)\left(\frac{2\delta_{jp}}{D} - \frac{4x_p x_j}{D^2}\right) + \frac{16x_i x_j(1-x_n)^2}{D^4}$$

$$= \frac{4\delta_{ij}}{D^2} - \frac{16x_i x_j}{D^3} + \frac{16(|x|^2-x_n^2)x_i x_j}{D^4} + \frac{16x_i x_j(1-x_n)^2}{D^4}$$

$$= \frac{4\delta_{ij}}{D^2} + \frac{16x_i x_j}{D^4}\underbrace{\left(-D + |x|^2 - x_n^2 + (1-x_n)^2\right)}_{=0}$$

$$= \frac{4\delta_{ij}}{D^2}$$

and $$\partial_i F \cdot \partial_n F = \sum_{p=1}^{n-1}\left(\frac{2\delta_{ip}}{D} - \frac{4x_p x_i}{D^2}\right)\frac{4x_p(1-x_n)}{D^2} + \frac{4x_i(x_n-1)}{D^2}\frac{2}{D^2}\left(-(|x|^2-x_n^2) + (1-x_n)^2\right)$$

$$= \frac{8x_i(1-x_n)}{D^3} - \frac{16(|x|^2-x_n^2)x_i(1-x_n)}{D^4} + \frac{8x_i(x_n-1)}{D^4}\left(-(|x|^2-x_n^2)+(1-x_n)^2\right)$$

$$= \frac{8x_i(1-x_n)}{D^4}\left[D - 2(|x|^2-x_n^2) - \left(-(|x|^2-x_n^2)+(1-x_n)^2\right)\right]$$

$$= 0$$

and $\partial_n F \cdot \partial_n F = \dfrac{16(|x|^2-x_n^2)(1-x_n)^2}{D^4} + \dfrac{4}{D^4}\left[(1-x_n)^2-(|x|^2-x_n^2)\right]^2$

$$= \frac{4}{D^2}$$

therefore: $(F^* g_{upper})(\partial_i,\partial_j) = \dfrac{1}{F_n^2}\,\partial_i F\cdot\partial_j F$

$$= \frac{4\delta_{ij}}{(1-|x|^2)^2} = g_{ball}(\partial_i,\partial_j)$$

$$(F^* g_{upper})(\partial_i,\partial_n) = \frac{1}{F_n^2}\,\partial_i F\cdot\partial_n F = 0 = g_{ball}(\partial_i,\partial_n)$$

$$(F^* g_{upper})(\partial_n,\partial_n) = \frac{1}{F_n^2}\,\partial_n F\cdot\partial_n F = \frac{4}{(1-|x|^2)^2} = g_{ball}(\partial_n,\partial_n)$$

Q3. assume first $M \underset{\text{diffeo}}{\cong} \mathbb{R}$

$\Rightarrow (M, g)$ isometric to $(\mathbb{R}, f^2 dx^2)$ for some $f > 0$ smooth

define $y(x) = \int_0^x f(t)\,dt$, $a = y(-\infty)$, $b = y(\infty)$

$\Rightarrow$ y gives diffeom $\mathbb{R} \longrightarrow (a, b)$ (since y strictly increasing)

and $f(x)^2 dx^2 = f(x)^2 \left(\frac{dx}{dy}\right)^2 dy^2$

$= f(x)^2 \frac{1}{f(x)^2} dy^2 = dy^2$

$\Rightarrow$ y gives isometry $(\mathbb{R}, f^2 dx^2) \longrightarrow ((a,b), dy^2)$

if $M \underset{\text{diff}}{\cong} S^1$ $\Rightarrow$ let $\tilde{M} \xrightarrow[\pi]{} M$ be universal cover

and let $\tilde{g} = \pi^* g$

then $(\tilde{M}, \tilde{g})$ isometric to $(\mathbb{R}, f^2 dx^2)$ for $f(x) =$ 1-periodic

$y(x)$ as above gives isometry $(\mathbb{R}, f^2 dx^2) \longrightarrow (\mathbb{R}, dy^2)$

and $y(x + n) = y(x) + Ln$ for any $n \in \mathbb{Z}$

and $L = \int_0^1 f(t)\,dt$

$\Rightarrow$ y descends to isometry $\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/L\mathbb{Z}$

Q4 let $T^2 = \left(\mathbb{R}^2 / (2\pi\mathbb{Z})^2 , d\theta^2 + d\varphi^2 \right)$ = flat torus

define $F(\theta, \varphi) = (e^{i\theta}, e^{i\varphi}) \subset \mathbb{C} \times \mathbb{C}$, clearly an embedding

then $F^* g_{eucl} = |\partial_\theta F|^2 d\theta^2 + 2\, \partial_\theta F \cdot \partial_\varphi F \, d\theta\, d\varphi + |\partial_\varphi F|^2 d\varphi^2$

$= d\theta^2 + d\varphi^2$

$= g_{eucl}$ $\Rightarrow$ F gives isometric embedding