Math 60670 Final Exam

You may quote any result we proved in class. Email me your solution by Wednesday May 6.

Q1. Let (M^n, g) be a smooth, connected, *non-complete* manifold without boundary. Endow M with any Riemannian metric g, so M has a metric space structure d_g. Let $\overline{M}$ be the metric space closure of M, so that $M \subset \overline{M}$ and $(\overline{M}, d_g)$ becomes a complete metric space. Write $d(x, \overline{M} \setminus M) := \inf_{y \in \overline{M} \setminus M} d_g(x, y)$ for the distance from x to the added points (the "boundary" of M). Note that $\overline{M} \setminus M \neq \emptyset$ by our non-completeness assumption.

A. Prove there is a smooth function $b(x) : M \to \mathbb{R}$ which satisfies

$$\frac{1}{2} \min\{d(x, \overline{M} \setminus M), 1\} \leq b(x) \leq 2 \min\{d(x, \overline{M} \setminus M), 1\} \qquad \forall x \in M.$$

In particular, $b(x) > 0$ on M and $\to 0$ as $x \to \overline{M} \setminus M$. Hint: use a partition of unity, and reduce to smoothing out the distance function in a coordinate chart.

B. Define the new metric $\bar{g} = b^{-2} g$. Prove that $d_{\bar{g}}(x, \overline{M} \setminus M) = \infty$ for all $x \in M$, and use this to show that $(M, \bar{g})$ is complete. Deduce that every smooth connected manifold admits a complete Riemannian metric.

Q2. Let (M^2, g) be a 2-dimensional complete, simply-connected Riemannian manifold with non-positive sectional curvature. Fix $p \in M$, and let $A(r) = \text{area}_M(B_r(p))$, $L(r) = \text{length}_M(\partial B_r(p))$. By Cartan-Hadamard's theorem both these functions are smooth for $r > 0$.

A. Prove that $A'(r) = L(r)$, $L'(r) = \int_{\partial B_r(p)} k ds$, where k is the geodesic curvature of $\partial B_r(p)$ w.r.t. the inwards normal.

B. Prove the Euclidean isoperimetric inequality: $4\pi A(r) \leq L(r)^2$. Hint: Consider the behavior of $f(r) = L(r)^2 - 4\pi A(r)$.

C. Show that if at some radius equality holds $4\pi A(r) = L(r)^2$, then $(B_r(p), g)$ is isometric to the flat Euclidean ball of radius r in $\mathbb{R}^2$.

Q3. Let (M^n, g) be complete and connected.

A. A geodesic line in M is a geodesic $\gamma : (-\infty, \infty) \to M$ which is minimizing on any finite interval. Prove that if M has positive sectional curvature, then M cannot have any geodesic line. Hint: use the stability inequality.

B. We say M has k ends if there is a sequence $K_1 \subset K_2 \subset \ldots$ of compact subests so that $M = \cup_i K_i$, $d(p, M \setminus K_i) \to \infty$ for any fixed $p \in M$, and $M \setminus K_i$ has k unbounded connected components for all $i >> 1$. Show that this is equivalent to the statement: for any $p \in M$, then $M \setminus \overline{B_r(p)}$ has k unbounded connected components for all r sufficiently large.

C. Show that if M has ≥ 2 ends, then M possesses a geodesic line. Deduce that if M has positive sectional curvature then M has at most one end.

Q4. **A.** Prove that a metric on $S^1 \times S^2$ cannot have only positive sectional curvature, or only non-positive sectional curvature.

B. Let $(M, g), (\bar{M}, \bar{g})$ be Riemannian manifolds, and $(M \times \bar{M}, g \oplus \bar{g})$ be the Riemannian product manifold. Prove that at every point $(p, \bar{p}) \in M \times \bar{M}$ there is a 2-plane $\subset T_p M \oplus T_{\bar{p}} \bar{M}$ for which the sectional curvature vanishes.

C. Prove that any if (M^n, g) is any closed Riemannian manifold, and if $S^2(r)$ denotes the Euclidean sphere of radius r, then $S^2(r) \times M$ has positive scalar curvature for small r.

Q5. Let $\Sigma^n \subset (M^{n+1}, g)$ be an smooth, connected, embedded hypersurface of dimension $n \geq 2$, with unit normal ν, and scalar-valued second fundamental form $h(X, Y) := \langle B(X, Y), \nu \rangle$. Assume Σ is umbilic, which means that $h = \lambda g|_\Sigma$ for some function $\lambda : \Sigma \to \mathbb{R}$.

A. Show that if M has constant sectional curvature, then λ must be constant. Hint: use the Codazzi equation.

B. Prove that if $\bar{g} = e^{2u} g$ for some smooth conformal factor e^{2u}, then Σ is umbilic in $(M, \bar{g})$.

C. Show that if $M^{n+1} = \mathbb{R}^{n+1}$, then Σ is a subset of either a sphere or a plane.