Q1 let $\bar{\nabla}$ = LC connection

define $A : \{\text{connections}\} \longrightarrow \{(2,1)\text{-tensors}\}$

$$\nabla \longmapsto A_\nabla(X,Y) = \nabla_X Y - \bar{\nabla}_X Y$$

check A well-defined

A_∇ trivially $\mathbb{R}$-linear, and C^∞ linear in X

check C^∞-linear in Y: $A_\nabla(X, fY)$

$$= \nabla_X(fY) - \bar{\nabla}_X(fY)$$

$$= X(f)Y - X(f)Y + fA_\nabla(X,Y) \checkmark$$

A_∇ bijective (since $\bar{\nabla} + A$ = connection for any $(2,1)$-tensor A)

choose E_i = ON frame @ p, write $A_\nabla(E_i, E_j) = A_{ij}^k E_k$

∇ torsion free @ p $\iff \nabla_X Y - \nabla_Y X - [X,Y] = 0$

$$= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

$$\iff \nabla_X Y - \bar{\nabla}_X Y - (\nabla_Y X - \bar{\nabla}_Y X) = 0 \qquad \forall X, Y$$

$$\iff A_\nabla(X,Y) = A_\nabla(Y,X) \qquad \forall X, Y$$

$$\iff A_{ij}^k = A_{ji}^k \qquad \forall i,j,k$$

∇ metric compatible @ p $\iff X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$

$$= X(g(Y,Z)) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z)$$

$$\iff g(\nabla_X Y - \bar{\nabla}_X Y, Z) + g(Y, \nabla_X Z - \bar{\nabla}_X Z) = 0 \qquad \forall X, Y, Z$$

$$\Leftrightarrow \quad g(A_D(X,Y),Z) + g(Y, A_D(X,Z)) = 0$$

$$\Leftrightarrow \quad A_{ij}{}^{k} = -A_{ik}{}^{j} \qquad \forall\, i,j,k$$

Q5 1. choose coords near p

$$\Rightarrow D\varphi_t Y\big|_p = Y^i\big|_p \frac{\partial \varphi_t^k(p)}{\partial x^k} \partial_k$$

so $$(\mathcal{L}_X T)(Y,Z) = \frac{\partial}{\partial t}\Big|_{t=0}\left(T_{ij}\Big|_{\varphi_t} Y^p\Big|_p \frac{\partial \varphi_t^i}{\partial x^p} Z^q\Big|_p \frac{\partial \varphi^j}{\partial x^q}\right)$$

$$= (\partial_t T_{ij}) Y^i Z^j + T_{ij} Y^p \partial_p X^k Z^j$$
$$+ T_{ij} Y^i Z^q \partial_q X^k$$

$$= \partial_t \left(T_{ij}\big|_{\varphi_t} Y^i\big|_{\varphi_t} Z^j\big|_{\varphi_t}\right)$$
$$- T_{ij} X^k (\partial_k Y^i) Z^j - T_{ij} Y^i X^k \partial_k Z^j)$$
$$+ T_{ij} Y^k (\partial_k X^i) Z^j + T_{ij} Y^i Z^k \partial_k X^j$$

$$= \partial_t (T(Y,Z)) - T([X,Y],Z)$$
$$- T(Y,[X,Z])$$

B. X = Killing $\iff \varphi_t^* g = g \quad \forall t, p$

$$\iff \frac{d}{dt} \varphi_t^* g = 0 \quad \forall t, p$$

$$\iff \frac{d}{dt} g_{\varphi_t(p)}(D\varphi_t Y, D\varphi_t Z) = 0 \quad \forall t, p, Y, Z$$

since $\varphi_{t+s} = \varphi_t \circ \varphi_s$

$$\iff \frac{d}{ds}\Big|_{s=0} g_{\varphi_s(\varphi_t(p))}(D\varphi_s \circ D\varphi_t Y, D\varphi_s \circ D\varphi_t Z) = 0 \quad \forall t$$
$$\forall Z, Y \; \forall p$$

$$\iff \left(\mathcal{L}_X g\big|_{\varphi_t(p)}\right)(D\varphi_t Y, D\varphi_t Z) = 0 \quad \forall Y, Z \quad \forall t \quad \forall p$$

$$\iff \mathcal{L}_X g\big|_p = 0 \quad \forall p \qquad \text{since } D\varphi_t \text{ non-singular}$$

φ_t diffeo —

$$\iff X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]) = 0$$

$$\forall X, Y, Z \in \mathfrak{X}(M)$$

$$\iff g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$- g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) = 0$$

$$\iff g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \quad \forall X, Y, Z \in \mathfrak{X}(M)$$

<u>C.</u> if $X = \text{grad } f$

then X killing $\iff g(\nabla_Y \text{grad } f, Z) + g(Y, \nabla_Z \text{grad } f) = 0$

$$\iff \nabla^2_{Y,Z} f + \nabla^2_{Z,Y} f = 0$$

$$\iff 2\nabla^2_{Y,Z} f = 0$$

<u>D.</u> γ geodesic $\implies \gamma'\langle \gamma', X\rangle = \langle \cancelto{0}{\nabla_{\gamma'}\gamma'}, X\rangle + \langle \gamma', \nabla_{\gamma'} X\rangle$

$$= \frac{1}{2}\left(\langle \gamma', \nabla_{\gamma'} X\rangle + \langle \nabla_{\gamma'} X, \gamma'\rangle\right)$$

$$= 0$$

Q: WLOG spse $M \setminus K_i$ has m components $\forall i$

choose $p_i, q_i \in M \setminus K_i$ lying in different ends

then $\mathrm{dist}(p_i, K_1) \to \infty$ and $\mathrm{dist}(q_i, K_1) \to \infty$

and if γ_i = minimizing geodesic $p_i \to q_i$

necessarily $\gamma_i \cap K_1 \neq \emptyset$ $\forall i$

$\Rightarrow$ WLOG $\gamma_i : [a_i, b_i] \to M$ PBAL and $\gamma_i(0) \in K_1$

now (after passing to subseq) $\gamma_i(0) \to p \in K_1$

and $\gamma_i'(0) \to v \in T_pM$ (since $|\gamma_i'(0)| = 1$)

$\Rightarrow$ by continuous dependence on initial conditions $\gamma_i \to \gamma_{p,v} =: \gamma$ in C^1_{loc}

i.e. $\forall T$, $\gamma_i|_{[-T,T]} \to \gamma_{p,v}|_{[-T,T]}$ in C^1

$\Rightarrow$ $L\gamma_i|_{[-T,T]} \to L\gamma$

$\parallel$

$d(\gamma_i(-T), \gamma_i(T))$ (for $i \gg 1$)

$\downarrow$

$d(\gamma(-T), \gamma(T))$ (since $\gamma_i(-T) \to \gamma(-T)$, etc)

$\Rightarrow$ $\gamma|_{[-T,T]}$ mmz $\forall T$ $\Rightarrow$ γ = geodesic line