Math 60670 Homework 8

Due Monday, April 14.

Problem 1: A Riemannian submanifold $M \subset (\bar{M}, \bar{g})$ (with the induced metric $g = \bar{g}|_{TM}$), is called totally geodesic if for every $p \in M$, $v \in T_pM$, the $\bar{g}$-geodesic γ with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$ lies in M. Show the following are equivalent:

A. M is totally geodesic,

B. Every g-geodesic in M is also a $\bar{g}$-geodesic in $\bar{M}$,

C. The second fundamental form of M vanishes.

Problem 2: Let $M \subset \mathbb{R}^3$ be the catenoid, which is the surface of revolution obtained by revolving the curve $x = \cosh z$ around the z-axis. Show that M has zero mean curvature, i.e. show the trace of the second fundamental form is zero.

Problem 3: A geodesic triangle in a Riemannian 2-manifold (M^2, g) is a domain Ω with piecewise-smooth boundary $\partial\Omega$ consisting of three geodesics meeting at three vertices. If M has constant Gauss curvature K, show that the sum of interior angles of any geodesic triangle is $\pi + KA$, where A is the area of Ω.

Problem 4: Let X, Y be smooth vector fields in a smooth manifold M, and let $\phi_t(x), \psi_s(x)$ be their respective flows, i.e. so that $\partial_t \phi_t(x) = X(\phi_t(x))$, and $\partial_s \psi_x(x) = Y(\psi_s(x))$. Suppose that $[X, Y] \equiv 0$.

A. Using the definition of Lie derivative, and the group property of flows (which is that $\phi_t \circ \phi_s = \phi_{t+s}$), show that $Y(\phi_t(x)) = D\phi_t Y(x)$. Similarly, deduce that $X(\psi_s(x)) = D\psi_s X(x)$.

B. Show that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$. Hint: consider the function $F(x, s, t) = (\phi_{-t} \circ \psi_{-s} \circ \phi_t \circ \psi_s)(x)$, and show that $F(x, s, t) \equiv x$.

C. Deduce there is a smooth function $f_{t,s}(x)$ with the property that $\partial_t f_{t,s}(x) = X(f_{t,s}(x))$ and $\partial_s f_{t,s}(x) = Y(f_{t,s}(x))$.