Q : let g, ∇ be induced metric, connection on M

take $p \in M$, $v \in T_pM$

let γ = g - geodesic s.t. $\gamma(0) = \bar\gamma(0) = p$, $\gamma'(0) = \bar\gamma'(0) = v$
$\bar\gamma$ = $\bar g$ - geodesic

M tot. geodesic $\Rightarrow$ $\bar\gamma$ lies in M
(A)
$\Rightarrow \bar\gamma' \in T_{\bar\gamma}M$

$\Rightarrow \nabla_{\bar\gamma'}\bar\gamma' = \pi(\bar\nabla_{\bar\gamma'}\bar\gamma') = 0$

$\Rightarrow \bar\gamma = \gamma$ by uniques of ODEs

$\Rightarrow$ every g-geodesic = $\bar g$ - geodesic (B)

since p, v arbitrary

spse $\gamma \equiv \bar\gamma$ $\forall$ choices of p, v (B)

$\Rightarrow \nabla_{\gamma'(0)}\gamma'(0) = \bar\nabla_{\bar\gamma'(0)}\bar\gamma'(0)$

$\Rightarrow B|_p(v,v) = \bar\nabla_{\bar\gamma'(0)}\bar\gamma'(0) - \nabla_{\bar\gamma'(0)}\gamma'(0) = 0$ (C)

$\Rightarrow B = 0 \;\forall p, v$

spse $B \equiv 0 \Rightarrow \bar\nabla_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + B(\gamma', \gamma')$
(C)
$= 0$

$\Rightarrow \gamma = \bar\gamma \Rightarrow \bar\gamma$ lies in $M \Rightarrow M$ tot. geodesic
(A)

Q2: catenoid: $r = \cosh(z) = \sqrt{x^2+y^2}$

$\hookrightarrow$ parametrization: $F(\theta, z) = (\cosh(z)\cos\theta, \cosh(z)\sin\theta, z)$
$\theta \in [0, 2\pi)$, $z \in \mathbb{R}$

$\Rightarrow \partial_\theta F = (-\cosh(z)\sin\theta, \cosh(z)\cos\theta, 0)$

$\partial_z F = (\sinh(z)\cos\theta, \sinh(z)\sin\theta, 1)$

$\Rightarrow g_{\theta\theta} = \partial_\theta F \cdot \partial_\theta F = \cosh^2(z)$

$g_{\theta z} = \partial_\theta F \cdot \partial_z F = 0$

$g_{zz} = \sinh^2(z) + 1 = \cosh^2(z)$

so $g_{ij} = \begin{array}{cc} \theta & z \\ \cosh^2(z) & 0 \\ 0 & \cosh^2(z) \end{array}$

choice of unit normal $= \nu = \dfrac{(-\cos\theta, -\sin\theta, \sinh(z))}{\sqrt{1+\sinh^2(z)}}$

scalar second fundamental form in coords $= h_{ij} = \partial^2_{ij} F \cdot \nu$

$\partial^2_\theta F = (-\cosh(z)\cos\theta, -\cosh(z)\sin\theta, 0)$

$\partial^2_{\theta z} F = (-\sinh(z)\sin\theta, \sinh(z)\cos\theta, 0)$

$\partial^2_{zz} F = (\cosh(z)\cos\theta, \cosh(z)\sin\theta, 0)$

$\Rightarrow h_{\theta\theta} = \partial^2_\theta F \cdot \nu = \dfrac{\cosh(z)}{\sqrt{1+\sinh^2(z)}} = 1$, $h_{\theta z} = 0$, $h_{zz} = \dfrac{-\cosh(z)}{\sqrt{1+\sinh^2(z)}} = -1$

$$\Rightarrow \quad h^i{}_j = g^{ip} h_{pj} = \frac{1}{\cosh^2(z)} \boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}$$

$\Rightarrow$ H = scalar mean curvature

$= \mathrm{tr}_g h$

$= 0$

Q3: recall Gauss-Bonnet says: if Ω curved polygon in (M,g)

$$\Rightarrow \int_{\Omega} K + \int_{\partial\Omega} k + \sum \varepsilon_i = 2\pi \chi(\Omega)$$

Gauss curvature

geodesic curvature of ∂

exterior angles

Euler characteristic

if Ω geodesic triangle $\Rightarrow \chi = V - E + F$

$= 3 - 3 + 1$

$= 1$

and $\partial\Omega$ has zero curvature

and $\underbrace{\sum \varepsilon_i}_{\text{ext. angles}} = 3\pi - \underbrace{\sum \theta_i}_{\text{interior angles}}$

since $K = \text{const}$

$$\Rightarrow K|\Omega| + 3\pi - \sum \theta_i = 2\pi$$

$$\Rightarrow \sum \theta_i = \pi + K|\Omega|$$

Q4. a. we have $[X,Y]\Big|_p = \lim_{t\to 0} \frac{D\varphi_{-t}Y(\varphi_t(p)) - Y(p)}{t}$

$$= \frac{d}{dt}\Big|_{t=0} D\varphi_{-t}Y(\varphi_t(p)) = 0 \quad \forall p$$

now since $\varphi_{s+t} = \varphi_s \circ \varphi_t \implies D\varphi_{s+t} = D\varphi_s \circ D\varphi_t$

so $\frac{d}{dt}\Big|_{t=s} D\varphi_{-t}(Y(\varphi_t(p))) = \lim_{t\to 0} \frac{D\varphi_{-t-s}Y(\varphi_{t+s}(p)) - D\varphi_{-s}Y(\varphi_s(p))}{t}$

$$= D\varphi_{-s}\left(\lim_{t\to 0} \frac{D\varphi_{-t}Y(\varphi_t(\varphi_s(p))) - Y(\varphi_s(p))}{t}\right)$$

$$= D\varphi_{-s}\left([X,Y]\Big|_{\varphi_s(p)}\right)$$

$$= 0$$

$\implies D\varphi_{-t}Y(\varphi_t(p)) = \text{const in } t$

$$= D\varphi_0 Y(\varphi_0(p))$$

$$= Y(p)$$

$\implies D\varphi_t Y\Big|_p = Y\Big|_{\varphi_t(p)}$

and by same reason, $D\varphi_t X\Big|_p = X\Big|_{\varphi_t(p)}$, $D\psi_t X\Big|_p = X\Big|_{\psi_t(p)}$, $D\psi_t Y\Big|_p = Y\Big|_{\psi_t(p)}$

B: consider function $F = \varphi_{-t} \circ \psi_{-s} \circ \varphi_t \circ \psi_s$

$$\Rightarrow \frac{\partial}{\partial s} F \Big|_p = D\varphi_{-t} Y \Big|_{\psi_{-s} \circ \varphi_t \circ \psi_s(p)} + \underbrace{D\varphi_{-t} \circ D\psi_{-s} \circ D\varphi_t Y \Big|_{\psi_s(p)}}_{\shortparallel}$$

$$D\varphi_{-t} \circ D\psi_{-s} Y \Big|_{\varphi_t \circ \psi_s}$$

$$= D\varphi_{-t} Y \Big|_{\psi_{-s} \circ \varphi_t \circ \psi_s}$$

$$\Rightarrow \frac{\partial}{\partial s} F \Big|_p = 0 \quad \forall\, t, s, p$$

and similarly $\frac{\partial}{\partial t} F \Big|_p = 0$. $\Rightarrow \varphi_t \circ \psi_s = \psi_s \circ \varphi_t$

C. define $f_{t,s}(x) = (\varphi_t \circ \psi_s)(x) \equiv (\psi_s \circ \varphi_t)(x)$

then $\partial_t f = \partial_t (\varphi_t \circ \psi_s) = X|_f$

$\partial_s f = \partial_s (\psi_s \circ \varphi_t) = Y|_f$