Q1 $\underline{A \Leftrightarrow B}$ ∇ compat with $g \Leftrightarrow Xg(Y,Z) = g(D_X Y, Z) + g(Y, D_X Z)$ $\forall X,Y,Z$

but LHS $= (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, D_X Z)$

so ∇ compat w/ $g \Leftrightarrow \nabla g = 0$

$\underline{B \Leftrightarrow C}$ recall that if $\gamma(t) = (x^1(t), \dots, x^n(t))$

then $\frac{DV}{dt} = (\dot{V}^k + \dot{x}^i V^j \Gamma_{ij}^k)\partial_k$

and $\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{ki}^p g_{pj} - \Gamma_{kj}^p g_{ip}$

$$\Rightarrow \frac{d}{dt} g(V,W) = \frac{d}{dt}(g_{ij} V^i W^j)$$

$$= (\dot{x}^k \partial_k g_{ij}) V^i W^j + g_{ij} \dot{V}^i W^j + g_{ij} V^i \dot{W}^j$$

$$= (\nabla_{\gamma'} g_{ij}) V^i W^j + g_{ij}(\dot{V}^i + \dot{x}^k V^q \Gamma_{kq}^i) W^j + g_{ij} V^i (\dot{W}^j + \dot{x}^k W^q \Gamma_{kq}^j)$$

$$= (\nabla_{\gamma'} g)(V,W) + g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)$$

$$= g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right) \quad \forall V, W, \gamma$$

$\Leftrightarrow (\nabla_{\gamma'} g)(V,W) = 0 \quad \forall V, W, \gamma$

$\Leftrightarrow \nabla g = 0$

$\underline{C \Rightarrow D}$ if V, W parallel $\Rightarrow \frac{d}{dt} g(V,W) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)$

$= 0$

D $\Rightarrow$ E   if $V(t), W(t)$ parallel transport along γ

then $g(V(t), W(t)) = g(V(0), W(0)) \Rightarrow V(0) \mapsto V(t)$

= isometry

note map is linear by uniqueness of parallel transport

↳ since $aV(t) + bW(t)$ is a parallel transport of $aV(0)+bW(0)$

$\Rightarrow$ must be the parallel transport

E $\Rightarrow$ C   let $E_1, \ldots, E_n$ = ON. basis for $T_{\gamma(0)}M$

↳ extend to $E_i(t) \in T_{\gamma(t)}M$ by parallel transport

$\Rightarrow$ if $V(t) = V^i(t) E_i(t)$ then $\frac{DV}{dt} = \dot{V}^i(t) E_i(t)$

by assumption, $\{E_i(t)\}$ ON basis for $T_{\gamma(t)}M$

$\Rightarrow g(E_i(t), E_j(t)) = \delta_{ij}$ = const in t

$$\Rightarrow \frac{d}{dt}\left(g(V,W)\right) = \frac{d}{dt}\left(V^i(t) W^j(t)\, g(E_i(t), E_j(t))\right)$$

$$= \delta_{ij} \dot{V}^i(t) W^j(t) + \delta_{ij} V^i(t) \dot{W}^j(t)$$

$$= g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)$$

Q3 A.

claim: if $X, Y \in \mathfrak{X}(M)$ then $\widetilde{\nabla}_{D\varphi(X)} D\varphi(Y) = D\varphi(\nabla_X Y)$

choose coords $f(x^1, \dots, x^n)$ $\mathcal{U} \subset \mathbb{R}^n \to M$ near p

$\Rightarrow (\varphi \circ f)(\tilde{x}^1, \dots, \tilde{x}^n) : \mathcal{U} \to \widetilde{M}$ coords on $\widetilde{M}$ near $\varphi(p)$

and if $F : \widetilde{M} \to \mathbb{R}$

$\hookrightarrow \widetilde{\partial}_i F = \partial_i (F \circ \varphi)$

write $X = X^i \partial_i$, $Y = Y^i \partial_i$

$\Rightarrow D\varphi(X)\big|_{\tilde{x}} = (X^i \circ \varphi^{-1})(\tilde{x})\, \widetilde{\partial}_i\big|_{\tilde{x}}$

since $\tilde{g}_{ij} = \langle \widetilde{\partial}_i, \widetilde{\partial}_j \rangle = \langle D\varphi(\partial_i), D\varphi(\partial_j) \rangle = g_{ij}$

$\Rightarrow \widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$

$\Rightarrow \widetilde{\nabla}_{D\varphi(X)} D\varphi(Y) = \widetilde{\nabla}_{X^i \circ \varphi^{-1} \widetilde{\partial}_i} (Y^j \circ \varphi^{-1} \widetilde{\partial}_j)$

$= X^i \circ \varphi^{-1}\, \widetilde{\partial}_i (Y^j \circ \varphi^{-1})\, \widetilde{\partial}_j + X^i \circ \varphi^{-1}\, Y^j \circ \varphi^{-1}\, \widetilde{\Gamma}_{ij}^k \widetilde{\partial}_k$

$= (X^i \circ \varphi^{-1})(\partial_i Y^j) \circ \varphi^{-1}\, \widetilde{\partial}_j + X^i \circ \varphi^{-1} (Y^j \circ \varphi^{-1})\, \Gamma_{ij}^k \widetilde{\partial}_k$

$= D\varphi\left(X^i \partial_i Y^j \partial_j + X^i Y^j \Gamma_{ij}^k \partial_k \right)$

$= D\varphi(\nabla_X Y)$

now $\exp_p(v)$ defined

$\Leftrightarrow$ $\exp_p(tv) = \gamma(t)$ = geodesic for $t \in [0,1]$

with $\gamma(0) = p$, $\gamma'(0) = v$

$\Leftrightarrow$ $\tilde{\gamma}(t) = \varphi(\gamma(t))$ = geodesic in $\tilde{M}$

with initial conditions $\tilde{\gamma}(0) = \varphi(p)$, $\tilde{\gamma}'(0) = D\varphi|_p v$

$\Leftrightarrow$ $\tilde{\gamma}(t) = \exp_{\varphi(p)}(t\, D\varphi|_p v)$ defined for $t \in [0,1]$

hence $\varphi(\exp_p(v)) = \exp_{\varphi(p)} D\varphi|_p v$, and both exist whenever either side exists

B. let $U = \{ x \in M$ s.t. $\varphi(x) = \tilde{\varphi}(x)$ and $D\varphi|_x = D\tilde{\varphi}|_x \}$

$p \in U \Rightarrow U \neq \emptyset$

U closed since $\varphi, \tilde{\varphi}$ smooth

if $q \in U \Rightarrow$ choose $\varepsilon > 0$ st $\exp_q : B_\varepsilon(0) \to M$ = diffeo

$\Rightarrow$ if $|v| < \varepsilon$

then $\varphi(\exp_q(v)) = \exp_{\varphi(q)} D\varphi|_q v$

$= \exp_{\varphi(q)} D\tilde{\varphi}|_q v$

$= \tilde{\varphi}(\exp_q(v))$

$\Rightarrow \varphi = \tilde{\varphi}$ on $B_\varepsilon(q)$ $\Rightarrow$ U open $\Rightarrow$ $U = M$

C. If φ = isometry $\Rightarrow$ length γ = length $\varphi\circ\gamma$

for any curve γ in M

$\Rightarrow d(p,q) = d(\varphi(p), \varphi(q)) \quad \forall p,q \in M$

spse φ = distance preserving

if $\gamma(t)$ = curve in M, $\gamma(0) = p$

$\Rightarrow d(\gamma(t), p) = |\exp_p^{-1}(\gamma(t))|$ for small t

$$\Rightarrow \frac{d(\gamma(t), p)}{t} = \left| \frac{\exp_p^{-1}(\gamma(t)) - \exp_p^{-1}(\gamma(0))}{t} \right| \quad (t>0)$$

$$\xrightarrow[t\to 0_+]{} |D\exp_p^{-1}|_p \, \gamma'(0)| = |\gamma'(0)|$$

So if $\gamma(t) = \exp_p(tv)$

$$\Rightarrow \frac{d(\varphi(\gamma(t)), \varphi(p))}{t} = \frac{d(\gamma(t), p)}{t}$$

$$\xrightarrow[t\to 0_+]{} |D\varphi|_p(\gamma'(0))| = |\gamma'(0)|$$

$\Rightarrow |D\varphi|_p(v)| = |v| \quad \forall p \in M, v \in T_pM$

now $|v+w|^2 = |D\varphi(v+w)|^2$

$= |D\varphi(v)|^2 + 2\langle D\varphi(v), D\varphi(w)\rangle + |D\varphi(w)|^2$

$= |v|^2 + 2\langle D\varphi(v), D\varphi(w)\rangle + |w|^2$

$= |v|^2 + 2\langle v, w\rangle + |w|^2$

$\Rightarrow \langle v, w\rangle = \langle D\varphi(v), D\varphi(w)\rangle$

Q1 A. from HW4, Q3, if $\varphi: M \to \bar{M}$ = isometry

then $\bar{\nabla}_{D\varphi(X)} D\varphi(Y) = D\varphi(\nabla_X Y)$

if γ = geodesic in M

$\Rightarrow \bar{\gamma} = \varphi \circ \gamma$ satisfies $\bar{\nabla}_{\bar{\gamma}'} \bar{\gamma}' = \bar{\nabla}_{D\varphi(\gamma')} D\varphi(\gamma')$

$= D\varphi(\nabla_{\gamma'} \gamma') = 0$

$\Rightarrow \bar{\gamma}$ = geodesic in $\bar{M}$

B. let $\gamma(t): I \to \mathbb{H}^2$ be geodesic with initial conditions

$$\begin{cases} \gamma(0) = (0,1) \\ \gamma'(0) = (0,1) \end{cases}$$

$R(x,y) = (-x,y)$ = isometry of $\mathbb{H}^2$

and $R(0,1) = (0,1)$, $DR\big|_{(0,1)}(0,1) = (0,1)$

$\Rightarrow R$ preserves γ

but $R(x,y) = (x,y) \iff x = 0$

$\Rightarrow \gamma(t) = (0, y(t))$

since $|\gamma'| = 1 \Rightarrow \dfrac{y'^2}{y^2} = 1$

$\Rightarrow y' = y \Rightarrow y(t) = e^t$

so $\gamma(t) = (0, e^t)$ exists $\forall t \in \mathbb{R}$

let $A = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \in SL(2, \mathbb{R}) \implies z \mapsto A \cdot z$ = isometry by HW2 Q3

$\Rightarrow \tilde{\gamma}(t) = A \cdot \gamma(t)$ = geodesic

now $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t)) = \dfrac{2ie^t + 1}{ie^t + 1}$

$$= \left(\frac{2e^{2t} + 1}{e^{2t} + 1}, \frac{e^t}{e^{2t} + 1} \right)$$

so $\tilde{y} > 0$, $\tilde{\gamma} \to (2, 0)$ $t \to \infty$

$\to (1, 0)$ $t \to -\infty$

and $(\tilde{x}(t) - \frac{3}{2})^2 + \tilde{y}(t)^2$

$$= \frac{(2e^{2t} + 1 - \frac{3}{2}(e^{2t} + 1))^2 + e^{2t}}{(e^{2t} + 1)^2}$$

$$= \frac{1}{4}$$

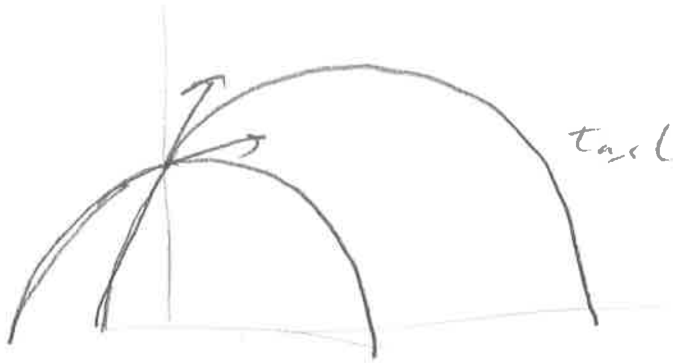
$\Rightarrow \tilde{\gamma}$ traces out half-circle $C = \{ (x - \frac{3}{2})^2 + y^2 = (\frac{1}{2})^2, y > 0 \}$

observe $\tau_{a,r}(x, y) = (a + rx, ry)$ = isometry of $\mathbb{H}^2$ $\forall a \in \mathbb{R}, r > 0$

↳ given any unit vector $v = (\alpha, \beta)$ with $\alpha \neq 0$

$\Rightarrow \exists\ a, r$ st $\tau_{a,r}(C)$ passes through $(0,1)$

and tangent space $T_{(0,1)}\tau_{a,r}(C)$ spanned by v



$\Rightarrow$ if $\gamma_v(t) = \tau_{a,r}\hat{\gamma}(t)$

then $\exists\ t_0$ st $\gamma_v(t_0) = (0,1)$, $\gamma_v'(t_0) = \pm v$

by uniqueness of geodesics, every geodesic passing through $(0,1)$

is either vertical half-line $\{x=0, y>0\}$

or half-circle $\tau_{a,r}(C)$ for some $a, r>0$

since $\tau_{a,r}$ acts transitively on $\mathbb{H}^2$

$\Rightarrow$ every geodesic takes the form $\tau_{a,r}\{x=0, y>0\}$

$= \{x=a, y>0\}$

or $\tau_{a,r}(C)$

<u>C.</u> from part B, if $\sigma(t)$ - geodesic in $\mathbb{H}^2$ with $|\sigma'|=1$

$\Rightarrow \exists$ isometry $\varphi: \mathbb{H}^2 \to \mathbb{H}^2$ st $(\varphi\circ\sigma)(t) = \gamma(t) = (0, e^t)$

since $\gamma(t)$ exists $\forall t \Rightarrow \sigma(t)$ exists $\forall t$