Math 60670 Homework 5

Due March 3.

Problem 1: A. The Laplacian of a function f is defined to be the trace of the Hessian: $\Delta f = tr_g(\nabla^2 f)$. Show that one can alternatively write

$$\Delta f = \mathrm{div}(\mathrm{grad} f),$$

and hence in coordinates

$$\Delta f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j f), \quad g = \det g_{ij}.$$

B. A function $u \in C^2(M)$ is called harmonic if $\Delta u = 0$. If (M, g) is compact with no boundary, prove that any harmonic function is necessarily constant. Hint: Use HW4 Problem 3.

Problem 2: A) For v in some interval I, let $(r(v), z(v))$ be a smooth, regular curve in the $r - z$ plane, with $r > 0$. Show that

$$F(\theta, v) = (r(v)\cos(\theta), r(v)\sin(\theta), z(v)), \quad t \in I, \theta \in \mathbb{R}.$$

is a well-defined immersion. The image of F is the surface of revolution obtained by rotating the curve $(r(v), z(v))$ about the z-axis. The lines corresponding to $\theta = const$, $v = const$ are called the meridians and parallels (respectively).

B) Show that the induced metric $F^* g_{eucl}$ in (θ, v) coordinates is given by

$$g_{11} = r^2, \quad g_{12} = 0, \quad g_{22} = r'^2 + z'^2.$$

C) Show that that $\gamma = (\theta(t), v(t))$ is a geodesic if and only if

$$\ddot{\theta} + \frac{2rr'}{r^2}\dot{\theta}\dot{v} = 0 \tag{1}$$

$$\ddot{v} - \frac{rr'}{r'^2 + z'^2}\dot{\theta}^2 + \frac{r'r'' + z'z''}{r'^2 + z'^2}\dot{v}^2 = 0. \tag{2}$$

D) Deduce meridians are always geodesics. When is a parallel geodesic?

E) Show that equations (1), (2) have the following "first order" interpretation: (2) is (except for meridians, parallels) equivalent to the fact that the "energy" $|\gamma'(t)|^2$ is constant; (1) is equivalent to the "Clairaut's relation:"

$$r(v(t)) \cos \beta(t) = const,$$

where $\beta(t)$ is the angle made between $\gamma'(t)$ and the parallel intersecting $\gamma(t)$.

F) Use Clairaut's relation to show that a geodesic of the paraboloid $r(t) = t, z(t) = t^2$ which is not a parallel or meridian must intersect itself infinitely-many times.