Q1 A. choose normal coords @ p $\Rightarrow$ $\nabla^2_{\partial_i,\partial_j} f = \partial_i\partial_j f$ @ p

and $g_{ij}|_p = \delta_{ij}$

$\Rightarrow \Delta f = \sum_i \partial_i \partial_i f$ @ p

and $\operatorname{grad} f = \sum_i \partial_i f \, \partial_i$ @ p

and $\operatorname{div}(X = X^i \partial_i) = \sum_i \partial_i X^i$ @ p

$\Rightarrow \sum_i \partial_i \partial_i f = \Delta f = \operatorname{div}(\operatorname{grad} f)$ @ p

in general coords, $\operatorname{div}(X) = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}\, X^i)$

$\operatorname{grad} f = g^{ij} \partial_j f \partial_i$

$\Rightarrow \Delta f = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}\, g^{ij} \partial_j f)$

B. since M has no ∂ and compact

$\Rightarrow 0 = \int \operatorname{div}(u \operatorname{grad} u) = \int |\operatorname{grad} u|^2 + u \Delta u$

$= \int |\operatorname{grad} u|^2$

$\Rightarrow \operatorname{grad} u = 0$

$\Rightarrow u =$ (locally) constant

Q2: A/B. $F(\theta, v) = (r(v)\cos\theta,\ r(v)\sin\theta,\ z(v))$

$$\Rightarrow \partial_\theta = (-r\sin\theta,\ r\cos\theta,\ 0)$$

$$\partial_v = (r'\cos\theta,\ r'\sin\theta,\ z')$$

$$\Rightarrow g_{\theta\theta} = r^2,\quad g_{\theta v} = 0,\quad g_{vv} = r'^2 + z'^2$$

since $r'^2 + z'^2 \neq 0$ (curve is regular)

and $r > 0 \Rightarrow g = \text{non-degenerate}$

$\Rightarrow F = \text{immersion}$

C. compute Γ_{ij}^k:

$$\Gamma_{\theta\theta}^\theta = \tfrac{1}{2} g^{\theta\theta} \partial_\theta g_{\theta\theta} = 0$$

$$\Gamma_{\theta\theta}^v = \tfrac{1}{2} g^{vv} (\partial_\theta g_{\theta v} + \partial_\theta g_{\theta v} - \partial_v g_{\theta\theta})$$

$$= \frac{1}{2} \frac{1}{r'^2 + z'^2} (-2rr')$$

$$\Gamma_{\theta v}^\theta = \tfrac{1}{2} g^{\theta\theta} (\partial_\theta g_{v\theta} + \partial_v g_{\theta\theta} - \partial_\theta g_{\theta v})$$

$$= \frac{1}{2} \frac{1}{r^2} (2rr')$$

$$\Gamma_{\theta v}^v = \tfrac{1}{2} g^{vv} (\partial_\theta g_{vv} + \partial_v g_{\theta v} - \partial_v g_{\theta v}) = 0$$

$$\Gamma_{vv}^\theta = \tfrac{1}{2} g^{\theta\theta} (\partial_v g_{\theta v} + \partial_v g_{\theta v} - \partial_\theta g_{vv}) = 0$$

$$\Gamma^{v}_{vv} = \frac{1}{2} g^{vv} \left(\partial_v g_{vv} \right) = \frac{1}{2} \frac{1}{r'^2 + z'^2} \left(2 r' r'' + 2 z' z'' \right)$$

geodesic equation: $\ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma^{k}_{ij} = 0$

so $\gamma(t) = (\theta(t), v(t))$ = geodesic

$\iff$

① $\ddot{\theta} + 2 \dot{\theta} \dot{v} \Gamma^{\theta}_{\theta v} = 0$

$$\iff \boxed{\ddot{\theta} + 2 \dot{\theta} \dot{v} \frac{r'}{r}} = 0 \quad =: E_1$$

② $\ddot{v} + \dot{\theta}^2 \Gamma^{v}_{\theta\theta} + \dot{v}^2 \Gamma^{v}_{vv} = 0$

$$\iff \boxed{\ddot{v} + \dot{\theta}^2 \left(\frac{-r r'}{r'^2 + z'^2} \right) + \dot{v}^2 \left(\frac{r' r'' + z' z''}{r'^2 + z'^2} \right)} = 0 \quad =: E_2$$

D/E: observe that $|\gamma'|^2 = r^2 \dot{\theta}^2 + (r'^2 + z'^2) \dot{v}^2$

$$\Rightarrow \frac{d}{dt} |\gamma'|^2 = 2 r r' \dot{\theta}^2 \dot{v} + 2 r^2 \dot{\theta} \ddot{\theta} \qquad \pm 2 r r' \dot{\theta}^2 \dot{v}$$

$$+ 2 (r' r'' + z' z'') \dot{v}^3 + (r'^2 + z'^2) 2 \dot{v} \ddot{v}$$

$$= 2 r^2 \dot{\theta} \left(\ddot{\theta} + 2 \dot{\theta} \dot{v} \frac{r'}{r} \right)$$

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$$+ 2 (r'^2 + z'^2) \dot{v} \left[\frac{-r r'}{r'^2 + z'^2} \dot{\theta}^2 + \frac{r' r'' + z' z''}{r'^2 + z'^2} \dot{v}^2 + \ddot{v} \right]$$

$$= 2 r^2 \dot{\theta} E_1 + 2 (r'^2 + z'^2) \dot{v} E_2$$

now γ = meridian PBAL $\iff$ $\dot\theta = 0$ and $|\gamma'|^2 = 1$

$$|\gamma'|^2 = (r'^2 + z'^2)\dot{v}^2 = 1$$

$\Rightarrow$ ① trivially holds

and $0 = \frac{d}{dt}|\gamma'|^2 = 2(r'^2 + z'^2)\dot{v}\, E_2$

$\Rightarrow$ ② holds since $\dot{v} \neq 0$

$\Rightarrow$ γ = geodesic

γ = parallel PBAL $\iff$ $\dot{v} = 0$ and $|\gamma'|^2 = 1$

$$|\gamma'|^2 = r^2\dot\theta^2$$

$\Rightarrow$ ② holds since $0 = \frac{d}{dt}|\gamma'|^2 = 2r^2\dot\theta\, E_1$

and $\dot\theta \neq 0$

① holds $\iff$ $\dfrac{rr'\dot\theta^2}{r'^2 + z'^2} = 0$

$\iff$ $r' = 0$ since $\dot\theta \neq 0$

$\hookrightarrow$ so γ geodesic $\iff$ $r' = 0$

for general regular curve, $\cos\beta = \dfrac{\langle \gamma', \partial_u \rangle}{|\gamma'||\partial_u|}$

$$= \frac{r\dot\theta}{|\gamma'|}$$

$\Rightarrow r\cos\beta = \dfrac{r^2\dot\theta}{|\gamma'|}$

if γ = geodesic $\Rightarrow E_1 = E_2 = 0$

$\Rightarrow \frac{d}{dt}|\gamma'| = 0$ (so wlog $|\gamma'|=1$)

$\Rightarrow \frac{d}{dt}(r\cos\beta) = 2rr'\dot r\dot\theta + r^2\ddot\theta$

$= r^2 E_1$

$= 0$

conversely, if $\frac{d}{dt}|\gamma'| = \frac{d}{dt}(r\cos\beta) = 0$

$\Rightarrow$ wlog $|\gamma'| = 1$ and hence $E_1 = 0$ by $\circledast$

$\Rightarrow \dot r E_2 = 0$ by $\circledast$

$\Rightarrow$ if γ not parallel then γ geodesic

$\underline{E.}$ paraboloid: $F(\theta, r) = (r\cos\theta, r\sin\theta, r^2)$

$$\Rightarrow g = \begin{bmatrix} r^2 & 0 \\ 0 & 1+4r^2 \end{bmatrix}$$

take $\gamma(t)$ geodesic $\Rightarrow |\gamma'| = $ const and $r\cos\beta = r^2\dot\theta =$ const

$= (\theta(t), r(t))$

$\Rightarrow$ after replacing t with $\pm$(const) t

wlog $|\gamma'| = 1$ and $r^2\dot\theta = c > 0$

(since γ not meridian, $\dot{\theta} \neq 0 \Rightarrow c \neq 0$)

$$|\gamma'| = r^2 \dot{\theta}^2 + (1+4r^2)\dot{r}^2 = 1$$

$$\Rightarrow \frac{c^2}{r^2} + (1+4r^2)\dot{r}^2 = 1$$

$$\Rightarrow \dot{r}^2 = \frac{1 - \frac{c^2}{r^2}}{1+4r^2}$$ so necessarily $r \geq c$

$\Downarrow$

$\gamma(t)$ exists $\forall t$

(since $F(\{r \geq c\})$ = closed subset of $\mathbb{R}^3$, and $F\{r > \frac{c}{2}\}$ = smooth submanifold)

<u>claim 1:</u> $\theta(t) \longrightarrow \pm\infty$ as $t \longrightarrow \pm\infty$

we have $\dot{r}^2 \leq 1$

$$\Rightarrow r(t) \leq r(0) + |t|$$

$$\Rightarrow \dot{\theta} = \frac{c}{r^2} \geq \frac{c}{\sqrt{r(0)+|t|}}$$

$$\Rightarrow \theta(t) - \theta(s) \geq \int_s^t \frac{c}{\sqrt{r(0)+|t|}}\, dt \longrightarrow \infty$$ as $t \to \infty$ or $s \to -\infty$

claim 2: $\dot r$ changes sign

spse $\dot r > 0 \ \forall t$ (same idea if $\dot r < 0 \ \forall t$)

$\Rightarrow c < r \leq r(0) \quad \forall t \leq 0$

$\Rightarrow r\dot r \geq \frac{1}{c_2}\sqrt{r^2-c^2} \quad \forall t \leq 0,$ $\quad c_2 =$ const depending only on $r(0), c$

$$\Rightarrow \sqrt{r^2-c^2}\Big|_{r(t)}^{r(0)} \geq \frac{1}{c_2}(0-t) \quad \text{for } t < \infty$$

$$\Rightarrow \sqrt{r(t)^2-c^2} \leq \sqrt{r(0)^2-c^2} + \frac{1}{c}t \longrightarrow -\infty \text{ as } t \to -\infty$$

$\lightning$

choose t_+ st $\dot r(t_+) > 0$ $(\Rightarrow r(t_+) > c)$

let $T = \max T$ st $\dot r > 0$ on $[t_+, T)$

$\hookrightarrow r(t) \geq r(t_+) > c$ on $[t_+, T)$ $\Rightarrow \dot r(T) > 0$ if $T < \infty$

$\Rightarrow$ i.e. $T = \infty$

and $$\dot r(t) = \sqrt{\frac{1 - r^2/c^2}{1+4r^2}} \geq \frac{1}{c_1 r} \quad t \geq t_+$$

$c_1 =$ const depending only on $r(t_+), c$

$$\Rightarrow \frac{d}{dt}(r^2) \geq \frac{1}{c_1} \quad t \geq t_+$$

$$\Rightarrow r^2(t) \geq r(t_+) + \frac{t}{c_1} \longrightarrow \infty \text{ as } t \to \infty$$

likewise, if we choose t_- st $\dot r(t_-) < 0$

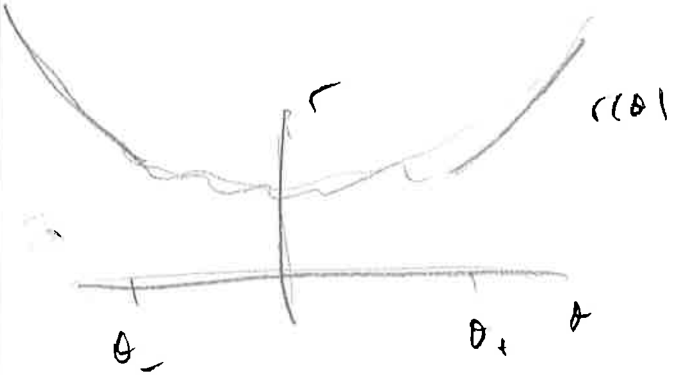
then $\dot r < 0 \quad \forall t < t_-$ and $r(t) \to \infty \quad t \to -\infty$

since $\dot\theta = \frac{c}{r^2} > 0 \implies$ can write $t = t(\theta) : \mathbb{R} \to \mathbb{R}$

$$\implies r = r(\theta)$$

$\hookrightarrow$ for $\theta \geq \theta_+ := \theta(t_+)$, $\frac{dr}{d\theta} = \frac{\dot r}{\dot\theta} > 0$, $r(\theta) \xrightarrow[\theta \to \infty]{} \infty$

$\theta \leq \theta_- := \theta(t_-)$, $\frac{dr}{d\theta} = \frac{\dot r}{\dot\theta} < 0$, $r(\theta) \xrightarrow[\theta \to -\infty]{} \infty$



so $\exists\, R_0$ and smooth functions $\theta_1, \theta_2 : [R_0, \infty) \to \mathbb{R}$

s.t. $\theta_1 \nearrow$ and $\theta_2 \searrow$

and $r(\theta_1(R)) = r(\theta_2(R)) = R$

$\implies \exists$ sequence $R_1 < R_2 < \dots$ s.t. $\theta_1(R_i) - \theta_2(R_i) \in 2\pi\mathbb{Z}$

and hence if $t_{1,i} = t(\theta_1(R_i))$

$t_{2,i} = t(\theta_2(R_i))$

then $\gamma(t_{1,i}) = \gamma(t_{2,i})$, $t_{1,i} - t_{2,i} \xrightarrow[i \to \infty]{} \infty$

$\implies \infty$-many intersections