Q1 choose coords x^i near p st $\nabla \partial_i|_p = 0$

since both sides tensorial,

ETS: $d\omega(\partial_i, \partial_j) = (\nabla_i \omega)(\partial_j) - (\nabla_j \omega)(\partial_i)$

$\omega = \omega_i dx^i \Rightarrow d\omega = \partial_i \omega_j dx^i \wedge dx^j$

so $d\omega(\partial_i, \partial_j) = \partial_i \omega_j - \partial_j \omega_i$

$$\underline{@p} \quad = \nabla_i \omega_j - \nabla_j \omega_i$$

$$= (\nabla_i \omega)(\partial_j) - (\nabla_j \omega)(\partial_i)$$

✓

Q1

Let E_i = ON frame @ p

parallel transport along γ $\longrightarrow$ get $E_i(t)$

$Y(\gamma(t)) = Y^i(t)\, E_i(t)$ for $Y^i(t) = \langle Y(\gamma(t)), E_i \rangle$

$P_{\gamma, Y(\gamma(t))}(s) = Y^i(t)\, E_i(s)$ since $\frac{D}{ds} P(s) = Y^i(t) \frac{DE_i}{ds} = 0$

$\Rightarrow P(t) = Y(\gamma(t))$

so $\frac{1}{t}\left(P_{\gamma, Y(\gamma(t))}(0) - Y(0)\right) = \frac{1}{t}\left(Y^i(t) - Y^i(0)\right) E_i(0)$

$\underset{t \to 0}{\longrightarrow} \dot{Y}^i(0)\, E_i(0)$

OTOH: $\nabla_X Y\Big|_p = \frac{D}{dt}\Big|_0 Y(t) = \dot{Y}^i(0)\, E_i + Y^i(0) \frac{DE_i(0)}{dt} = \dot{Y}^i(0)\, E_i(0)$ ✓

Q3

A. In Stokes' thm, orientation on ∂M induced by orientation on M by the rule: $e_1, \ldots, e_{n-1}$ positively oriented in $T_x \partial M \iff N, e_1, \ldots, e_{n-1}$ positively oriented in $T_x M$

↳ dV_M induces "compatible" volume form $dV_{\partial M}$

by $dV_{\partial M}(V_1, \ldots, V_{n-1}) := dV(N, V_1, \ldots, V_{n-1})$

now by Stokes' thm:
$$\int_M \operatorname{div}(X)\, dV_M = \int_M d(\iota_X dV_M)$$
$$= \int_{\partial M} \iota_X dV_M$$
$$= \int_{\partial M} \langle X, N \rangle \, dV_{\partial M}$$

(note that $dV_M(e_i, \cdot)\big|_{\partial M} = 0$ if $e_i \in T_x \partial M$ by anti-symmetry)

B. choose coords x^i near p st $x^i(0) = p$, $g_{ij}(0) = \delta_{ij}$
$$\partial_k g_{ij}\big|_0 = 0$$

recall that $\det(I + \varepsilon A) = 1 + \varepsilon \operatorname{tr} A + O(\varepsilon^2)$

for $X = X^i \partial_i$, $\operatorname{div}(X)\, dV \Big|_p = d(\iota_X\, dV)\Big|_p$

$$= d\left(\sqrt{\det g_{pq}}\ \sum_i X^i (-1)^{i+1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n\right)\Bigg|_{x=0}$$

omitted

$$= \frac{\partial}{\partial x^i}\Big|_0 \left(\sqrt{\det g_{pq}}\ X^i\right) dx^1 \wedge \cdots \wedge dx^n$$

$$= \left[\frac{1}{2\sqrt{\det \delta_{pq}}} \underbrace{\operatorname{tr}\left(\partial_{x^i} g_{pq}\big|_0\right)}_{=0} + \sqrt{\det \delta_{pq}}\ \partial_i X^i\Big|_0\right] dx^1 \wedge \cdots \wedge dx^n$$

$$= \partial_i X^i\Big|_0\, dx^1 \wedge \cdots \wedge dx^n$$

so $\operatorname{div}(X)\Big|_0 = \partial_i X^i\Big|_0$ in these coords

we compute: $\operatorname{div}(\varphi X)\Big|_0 = \partial_i(\varphi X^i) = (\partial_i \varphi) X^i + \varphi\, \partial_i X^i$

$$= \langle \operatorname{grad} \varphi, X\rangle\Big|_0 + \varphi \operatorname{div}(X)\Big|_0$$

now LHS and RHS independent of coord system

$\Rightarrow \operatorname{div}(\varphi X) = \langle \operatorname{grad} \varphi, X\rangle + \varphi \operatorname{div}(X) \quad \forall p$

C. for $p \in M \setminus \partial M$, take $\varphi \in C_c^\infty(M)$ sptd in some coord chart near p

then by A, B: $$0 = \int_M \mathrm{div}(\varphi X)\, dV = \int \langle \mathrm{grad}\, \varphi, X \rangle + \varphi\, \mathrm{div}(X)\, dV$$

$\Rightarrow$ in coords: $$\int_{\mathbb{R}^n} \varphi\, \mathrm{div}(X) \sqrt{g}\, dx^1 \cdots dx^n$$

$$= \int_{\mathbb{R}^n} - g_{ij}\, g^{ip} (\partial_p \varphi) X^j \sqrt{g}\, dx^1 \cdots dx^n$$

$g = \det(g_{pq})$

$$= - \int (\partial_j \varphi) X^j \sqrt{g}\, dx^1 \cdots dx^n$$

$$= \int \varphi\, \partial_j (X^j \sqrt{g})\, dx^1 \cdots dx^n$$

since φ arbitrary $\Rightarrow$ $$\mathrm{div}(X) = \frac{1}{\sqrt{g}} \partial_j (\sqrt{g}\, X^j)$$

D. recall that if x^i coords st $x^i(0) = p$, $g_{ij}(0) = \delta_{ij}$, $\partial_k g_{ij}|_0 = 0$

then $$\mathrm{div}(X)\big|_p = \partial_i X^i \big|_p$$

but $$\nabla_i X^i \big|_p = \partial_i X^i \big|_p \quad \text{since } \Gamma_{ij}^k(0) = 0$$

$$\Rightarrow \mathrm{div}(X)\big|_p = \partial_i X^i \big|_p = \nabla_i X^i \big|_p = \mathrm{tr}(\nabla X)\big|_p$$