Math 60670 Homework 3

Due February 17.

Problem 1: Show there are vector fields X_1, X_2, Y on $\mathbb{R}^2$, such that on the x^1-axis $X_1 = X_2 = (1,0)$ and $Y = (0,1)$, but such that the Lie derivatives $\mathcal{L}_{X_1}Y \neq \mathcal{L}_{X_2}Y$. (So, one cannot use the Lie derivative to define a reasonable notion of "derivative of vector field along a curve").

Problem 2: The torsion T of a linear connection ∇ is defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, and ∇ is called torsion free if $T \equiv 0$.

A) Show that the torsion is a $(1,2)$-tensor.

B) Show that the Euclidean connection on $\mathbb{R}^n$ is torsion free.

C) Prove that a linear connection ∇ is torsion-free if and only if the Christoffel symbols Γ_{ij}^k in any *coordinate frame* are symmetric in i and j, that is $\Gamma_{ij}^k = \Gamma_{ji}^k$. (Caveat: this is not necessarily true in a non-coordinate frame).

D) Prove that ∇ is torsion free if and only if the covariant Hessian $\nabla^2 u$ of any $u \in C^\infty(M)$ is a symmetric $(0,2)$-tensor.

Problem 3: Let $\mathbf{U}^2$ denote the hyperbolic plane, i.e. the upper half-plane in $\mathbb{R}^2$ with metric $h = (dx^2 + dy^2)/y^2$. Let $\mathrm{SL}(2,\mathbb{R})$ denote the group of 2×2 real matrices of determinant 1, and define an action of $A \in \mathrm{SL}(2,\mathbb{R})$ on points $z = x + iy \in \mathbf{U}^2 \subset \mathbb{C}$ by

$$A \cdot z = \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

Show this defines a smooth action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbf{U}^2$ by isometries of the hyperbolic metric.