Math 60670 Homework 2

Due February 10.

Problem 1: Let $\phi : M \to \bar{M}$ be a smooth map, X, Y vector fields on M, $\bar{X}, \bar{Y}$ vector fields on $\bar{M}$, and suppose $\bar{X}|_{\phi(p)} = D\phi(X|_p)$, $\bar{Y}|_{\phi(p)} = D\phi(Y|_p)$ for every $p \in M$. Show that $[\bar{X}, \bar{Y}]|_{\phi(p)} = D\phi([X, Y]|_p)$. Hint: First show that if $f \in C^\infty(\bar{M})$ then $\bar{X}(f)|_{\phi(p)} = X(f \circ \phi)|_p$ and $\bar{Y}(f)|_{\phi(p)} = Y(f \circ \phi)|_p$.

Problem 2: Let V be an n-dimensional vector spaces. Prove that the the space $(1, 1)$-tensors on V is naturally (i.e. independent of basis) isomorphic to the space of endomorphisms of V (i.e. the space of linear maps $V \to V$).

Problem 3: Let (x^i), (y^α) be local coordinates defined in some $U \subset M$. Suppose A is a $(1, 2)$-tensor field which in the x-coordinate system can be expressed as

$$A = A^i_{jk}(x)\partial_{x^i} \otimes dx^j \otimes dx^k.$$

Show that in the y-coordinate system the components of A are

$$A^a_{bc}(y = y(x)) = \frac{\partial y^a}{\partial x^i} \frac{\partial x^j}{\partial y^b} \frac{\partial x^k}{\partial y^c} A^i_{jk}(x).$$

Use this to show explicitly that the result of contracting the i, j indices together is independent of choice of coordinates.

Problem 4: Show that T is a smooth (k, l)-tensor field on M if and only if T is a smooth, $\mathbb{R}$-multilinear function from k 1-forms and l vector fields to $\mathbb{R}$, which is also multilinear over $C^\infty(M)$. By "smooth" we mean that if $X_1, \ldots, X_l \in \mathcal{X}(M)$, $\omega_1, \ldots, \omega_k \in \mathcal{X}^*(M)$, then $T(\omega_1, \ldots, \omega_l, X_1, \ldots, X_k) \in C^\infty(M)$.

Problem 5: Let (M, g) be an oriented Riemannian n-manifold, and let $(x^1, \ldots, x^n)$ be coordinates compatible with the orientation in the sense that $(\partial_{x^1}, \ldots, \partial_{x^n})$ is a positively-oriented basis. Show that the volume form $dV = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$.