Q1 If $\gamma(t)$ = curve in M with $\gamma(0)=p$, $\gamma'(0)=X$

then $\left.\frac{d}{dt}\right|_{t=0} (f\circ\varphi)(\gamma(t)) = X(f\circ\varphi)|_p$

$$\left.\frac{d}{dt}\right|_{t=0} f(\varphi(\gamma(t))) = (D\varphi|_p X)(f)|_{\varphi(p)} \quad \text{since } \left.\frac{d}{dt}\right|_{t=0}\varphi(\gamma(t)) = D\varphi|_p X$$

$$= \bar{X}(f)|_{\varphi(p)}$$

and similarly $Y(f\circ\varphi)|_p = \bar{Y}(f)|_{\varphi(p)}$

so $\bar{X}(\bar{Y}(f))|_q = X(\bar{Y}(f)\circ\varphi)|_{\varphi^{-1}(q)}$

$$= X(Y(f\circ\varphi))|_{\varphi^{-1}(q)}$$

$$\Rightarrow [\bar{X},\bar{Y}]f|_q = \bar{X}(\bar{Y}f)|_q - \bar{Y}(\bar{X}(f))|_q$$

$$= X(Y(f\circ\varphi))|_{\varphi^{-1}(q)} - Y(X(f\circ\varphi))|_{\varphi^{-1}(q)}$$

$$= [X,Y](f\circ\varphi)|_{\varphi^{-1}(q)}$$

$$= \left(D\varphi|_{\varphi^{-1}(q)}[X,Y]\right)(f)|_q$$

Q1 define $F : \mathrm{End}(V) \longrightarrow T^{(1,1)}(V)$

by $\ell \longmapsto F_\ell(v, \omega) = \omega(\ell(v))$

clearly linear

pick basis e_i for V, θ^i for V^* s.t. $\theta^i(e_j) = \delta_{ij}$

if $\ell(v^i e_i) = \ell^i_j v^j e_i$

then $F_\ell = \ell^i_j \theta^j \otimes e_i$

$\Rightarrow F =$ bijection $\Rightarrow$ linear isomorphism

Q2: let's show $T \in \mathcal{T}^{(1,1)}(M) \iff T : \mathfrak{X}(M) \otimes \mathfrak{X}^*(M) \longrightarrow C^\infty(M)$ smooth, linear over $C^\infty(M)$

general case is verbatim

$\Rightarrow$ trivial since in coords $T = T_i^j dx^i \otimes \partial_j$

$$\Rightarrow T(fX, g\omega) = f X^i g \omega_j T_i^j$$
$$= f g T(X, \omega)$$

for $X \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$
$\omega \in \mathfrak{X}^*(M)$

$\Leftarrow$ in coords we have $T(X^i \partial_i, \omega_j dx^j)$

$$= X^i \omega_j T(\partial_i, dx^j)$$

$$= \big(\underbrace{T(\partial_i, dx^j)\, dx^i \otimes \partial_j}_{=(1,1)\text{-tensor}}\big)(X, \omega)$$

Q3: by chain rule, $dx^i = \frac{\partial x^i}{\partial y^a} dx^a$

$$\frac{\partial}{\partial x^k} = \frac{\partial y^c}{\partial x^k}\frac{\partial}{\partial y^c}$$

$$\Rightarrow A^k_{ij}\, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

$$= \underbrace{A^k_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\frac{\partial y^c}{\partial x^k}}_{= A^c_{ab}}\, dy^a \otimes dy^b \otimes \frac{\partial}{\partial y^c}$$

$$(\mathrm{tr}\, A)_b = A^a_{ab} = A^k_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial y^a}{\partial x^k}\frac{\partial x^j}{\partial y^b} = A^i_{ij}\frac{\partial x^j}{\partial y^b}$$

$$= (\mathrm{tr}\, A)_j \frac{\partial x^j}{\partial y^b}$$

$\frac{\partial x^i}{\partial y^a}\frac{\partial y^a}{\partial x^k}$ " by chain rule $\frac{\partial x^i}{\partial x^k} = \delta_{ik}$

Q2: recall general property about n-forms on $\mathbb{R}^n$:

if $e_1 \ldots e_n$ = std basis, $\omega^1, \ldots, \omega^n$ = dual basis

$$\Rightarrow (\omega^1 \wedge \cdots \wedge \omega^n)(v_1, \ldots, v_n) = \det \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

let E_i = oriented, g-ON basis of T_pM

θ^i = dual basis

$$\Rightarrow dV\Big|_p = \theta^1 \wedge \cdots \wedge \theta^n$$

if x^i coords near $p \longrightarrow \partial_i = a_i^j E_j$

$$\Rightarrow dV(\partial_1, \ldots, \partial_n) = \det \begin{bmatrix} a_1^j & a_2^j & \cdots & a_n^j \end{bmatrix}$$

$$= \det A \quad \text{for } A_{ij} := a_j^i$$

now $g_{ij} = g(\partial_i, \partial_j) = a_i^p a_j^q \, g(e_p, e_q)$

$$= a_i^p a_j^q \delta_{pq}$$

$$= a_i^p a_j^p = (A^T A)_{ij}$$

$$\Rightarrow \det(g_{ij}) = (\det A)^2$$

$$\Rightarrow \det A = \sqrt{\det(g_{ij})} \quad \text{since } \partial_i \text{ positively-oriented}$$

(so $\det A > 0$)

$$\Rightarrow dV = \det A \, dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n$$