Math 60670 Homework 1

Due January 29.

Problem 1: Prove that the Lie bracket and Lie derivative coincide, i.e. if $X, Y \in \mathcal{X}(M)$, and $\phi_t(p)$ is the flow of X, then show

$$[X, Y](p) = \lim_{t \to 0} \frac{D\phi_{-t} Y(\phi_t(p)) - Y(p)}{t}.$$

Hint: If $X(p) \neq 0$, work in coords so that $X = \frac{\partial}{\partial x^1}$. If $X(p) = 0$, use that $\phi_t(p) = p$ to simplify the computation.

Problem 2: Using the map we defined in class or otherwise, show explicitly that the ball model and the half-space model of 2-dimensional hyperbolic space are isometric (in other words, find and verify an isometry between them).

Problem 3: Show that any smooth, connected, 1-dimensional Riemannian manifold is isometric to one of $(0, L)$, $(0, \infty)$, $\mathbb{R}$, $\mathbb{R}/L\mathbb{Z}$, for some L, where each of these manifolds inherits the metric $ds^2 = dx^2$ from $\mathbb{R}$. You may take for granted that any smooth, connected, 1-dimensional manifold is diffeomorphic to $(0, 1)$ or S^1 (though proving this is a nice exercise).

Problem 4: Construct an isometric embedding of a flat 2-torus into $\mathbb{R}^4$. Not for credit: Do you think this torus can be isometrically embedded into $\mathbb{R}^3$? You may find the “Nash embedding theorem” interesting.