Q1. first spse $X(p) \neq 0$

$\Rightarrow$ can choose coords near p so that $X \equiv \frac{\partial}{\partial x^1}$ near p

why? given coords $(y^1, \dots, y^n)$ near p so that $X = X^i \frac{\partial}{\partial y^i}$ with $X^1 \neq 0$

can define $(x^1, \dots, x^n) \mapsto \varphi_{x^1}(0, x^2, \dots, x^n)$

$\Rightarrow$ gives coords near p by inverse function thm

anyway in these coords, $[X,Y]|_p = XY - YX|_p$

$= (\partial_1 Y^i) \partial_i$

and $\varphi_t(x^1, \dots, x^n) = (x^1 + t, x^2, \dots, x^n)$

$\Rightarrow L_X Y|_p = \frac{d}{dt}\Big|_{t=0} D\varphi_{-t} Y(\varphi_t(p))$

$= \frac{d}{dt}\Big|_{t=0} Y^i(x^1+t, x^2, \dots, x^n) \partial_i$

$= (\partial_1 Y^i) \partial_i$

if $X(p) = 0$, could do options:

① (dirty trick) set $X_\varepsilon = X + \varepsilon \frac{\partial}{\partial x^1}$

$\Rightarrow [X_\varepsilon, Y] = L_{X_\varepsilon} Y$ by previous argument $\forall \varepsilon > 0$

$\Rightarrow [X,Y] = L_X Y$ (take $\varepsilon \to 0$)

(c) or note that $\varphi_t(p) = p$

$$\Rightarrow L_X Y = \frac{d}{dt}\Big|_{t=0} D\varphi_{-t} Y(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{\partial \varphi^i_{-t}}{\partial x^j}\Big|_p Y^j_p \, \partial_i\big|_p$$

$$= \left(\frac{\partial^2}{\partial x^j \partial t} \varphi^i_{-t}\Big|_p \right) Y^j \partial_i$$

$$= (-\partial_j X^i) Y^j \partial_i \qquad \text{since } \partial_t \varphi_{-t} = -X$$

and $$[X,Y]\big|_p = \cancel{XY}^{\,\to 0 \text{ @ } p} - YX$$

$$= -Y^j (\partial_j X^i) \partial_i$$

Q2. define $F(x_1, \dots, x_n) = \dfrac{(2x_1, \dots, 2x_{n-1}, 1-|x|^2)}{\underbrace{x_1^2 + \dots + x_{n-1}^2 + (1-x_n)^2}_{=: D}}$

want to show F = isometry $\left(B_1, g_{ball} = \dfrac{4\,dx^2}{(1-|x|^2)^2}\right)$

$\longrightarrow \left(\{x_n > 0\}, g_{upper} = \dfrac{dx^2}{x_n^2}\right)$

clearly F takes $B_1 \longrightarrow \{x_n > 0\}$ and is a smooth map

claim: F = diffeo $B_1 \longrightarrow \{x_n > 0\}$

define the (smooth) map $G(y_1, \dots, y_n) = \dfrac{(2y_1, \dots, 2y_{n-1}, |y|^2 - 1)}{y_1^2 + \dots + y_{n-1}^2 + (1+y_n)^2}$

ETS: $G \circ F = id_{B_1}$

$\hookrightarrow$ ETS: $\dfrac{(2F_1, \dots, 2F_{n-1}, |F|^2 - 1)}{|F|^2 - F_n^2 + (1+F_n)^2} = (x_1, \dots, x_n)$

now $|F|^2 - F_n^2 + (1+F_n)^2 = \dfrac{4(|x|^2 - x_n^2)}{D^2} + \dfrac{4(1-x_n)^2}{D^2} = \dfrac{4}{D}$

and $|F|^2 - 1 = \dfrac{4(|x|^2 - x_n^2) + (1-|x|^2)^2 - D^2}{D^2}$

$= \dfrac{4}{D^2}\left[|x|^2 - x_n^2 - |x|^2 + |x|^2 x_n + x_n - x_n^2\right]$

$= \dfrac{4x_n}{D}$

so $G \circ F = \dfrac{(2F_1, \dots, 2F_{n-1}, |F|^2 - 1)}{|F|^2 - F_n^2 + (1+F_n)^2} = \dfrac{\left(\dfrac{4x_1}{D}, \dots, \dfrac{4x_{n-1}}{D}, \dfrac{4x_n}{D}\right)}{4/D}$

$= (x_1, \dots, x_n)$ ✓

now WTS $F^* g_{upper} = g_{ball}$

computes for $i, p < n$

$$\partial_i F_p = \frac{2\delta_{ip}}{D} - \frac{4x_p x_i}{D^2}$$

$$\partial_i F_n = -\frac{2x_i}{D} - \frac{(1-|x|^2)2x_i}{D^2} = \frac{4x_i(x_n-1)}{D^2}$$

$$\partial_n F_p = \frac{2x_p}{D^2} \cdot 2(1-x_n) = \frac{4x_p(1-x_n)}{D^2}$$

$$\partial_n F_n = -\frac{2x_n}{D} + \frac{2(1-|x|^2)(1-x_n)}{D} = \frac{2}{D^2}\left[-(|x|^2-x_n^2) + (1-x_n)^2\right]$$

so: $$\partial_i F \cdot \partial_j F = \sum_{p=1}^{n-1}\left(\frac{2\delta_{ip}}{D} - \frac{4x_p x_i}{D^2}\right)\left(\frac{2\delta_{jp}}{D} - \frac{4x_p x_j}{D^2}\right) + \frac{16x_i x_j(1-x_n)^2}{D^4}$$

$$= \frac{4\delta_{ij}}{D^2} - \frac{16x_i x_j}{D^3} + \frac{16(|x|^2-x_n^2)x_i x_j}{D^4} + \frac{16x_i x_j(1-x_n)^2}{D^4}$$

$$= \frac{4\delta_{ij}}{D^2} + \frac{16x_i x_j}{D^4}\underbrace{\left(-D + |x|^2 - x_n^2 + (1-x_n)^2\right)}_{=0}$$

$$= \frac{4\delta_{ij}}{D^2}$$

and $$\partial_i F \cdot \partial_n F = \sum_{p=1}^{n-1}\left(\frac{2\delta_{ip}}{D} - \frac{4x_p x_i}{D^2}\right)\frac{4x_p(1-x_n)}{D^2} + \frac{4x_i(x_n-1)}{D^2}\frac{2}{D^2}\left(-(|x|^2-x_n^2) + (1-x_n)^2\right)$$

$$= \frac{8x_i(1-x_n)}{D^3} - \frac{16(|x|^2-x_n^2)x_i(1-x_n)}{D^4} + \frac{8x_i(x_n-1)}{D^4}\left(-(|x|^2-x_n^2) + (1-x_n)^2\right)$$

$$= \frac{8x_i(1-x_n)}{D^4}\left[D - 2(|x|^2-x_n^2) - \left(-(|x|^2-x_n^2) + (1-x_n)^2\right)\right]$$

$$= 0$$

and $$\partial_n F \cdot \partial_n F = \frac{16(|x|^2-x_n^2)(1-x_n)^2}{D^4} + \frac{4}{D^4}\left[(1-x_n)^2 - (|x|^2-x_n^2)\right]^2$$

$$= \frac{4}{D^2}$$

therefore: $$(F^* g_{upper})(\partial_i, \partial_j) = \frac{1}{F_n^2}\partial_i F \cdot \partial_j F$$

$$= \frac{4\delta_{ij}}{(1-|x|^2)^2} = g_{ball}(\partial_i, \partial_j)$$

$$(F^* g_{upper})(\partial_i, \partial_n) = \frac{1}{F_n^2}\partial_i F \cdot \partial_n F = 0 = g_{ball}(\partial_i, \partial_n)$$

$$(F^* g_{upper})(\partial_n, \partial_n) = \frac{1}{F_n^2}\partial_n F \cdot \partial_n F = \frac{4}{(1-|x|^2)^2} = g_{ball}(\partial_n, \partial_n)$$

Q3. assume first $M \underset{\text{diffeo}}{\cong} \mathbb{R}$

$\Rightarrow (M, g)$ isometric to $(\mathbb{R}, f^2 dx^2)$ for some $f > 0$ smooth

define $y(x) = \int_0^x f(t)\,dt$, $a = y(-\infty)$, $b = y(\infty)$

$\Rightarrow y$ gives diffeom $\mathbb{R} \longrightarrow (a, b)$ (since y strictly increasing)

and $$f(x)^2 dx^2 = f(x)^2 \left(\frac{dx}{dy}\right)^2 dy^2$$
$$= f(x)^2 \frac{1}{f(x)^2} dy^2 = dy^2$$

$\Rightarrow y$ gives isometry $(\mathbb{R}, f^2 dx^2) \longrightarrow ((a,b), dy^2)$

if $M \underset{\text{diff}}{\cong} S^1 \Rightarrow$ let $\tilde{M} \xrightarrow[\pi]{} M$ be universal cover
and let $\tilde{g} = \pi^* g$

then $(\tilde{M}, \tilde{g})$ isometric to $(\mathbb{R}, f^2 dx^2)$ for $f(x) =$ 1-periodic

$y(x)$ as above gives isometry $(\mathbb{R}, f^2 dx^2) \longrightarrow (\mathbb{R}, dy^2)$

and $y(x+n) = y(x) + Ln$ for any $n \in \mathbb{Z}$
and $L = \int_0^1 f(t)\,dt$

$\Rightarrow y$ descends to isometry $\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/L\mathbb{Z}$

Q4 let $T^2 = \left(\mathbb{R}^2 / (2\pi \mathbb{Z})^2 , d\theta^2 + d\varphi^2 \right)$ = flat torus

define $F(\theta, \varphi) = (e^{i\theta}, e^{i\varphi}) \subset \mathbb{C} \times \mathbb{C}$, clearly an embedding

then $F^* g_{eucl} = |\partial_\theta F|^2 d\theta^2 + 2 \partial_\theta F \cdot \partial_\varphi F \, d\theta \, d\varphi + |\partial_\varphi F|^2 d\varphi^2$

$= d\theta^2 + d\varphi^2$

$= g_{eucl}$ $\Rightarrow$ F gives isometric embedding