Math 60670 Final

Due the evening of Thursday May 8. Send your exam to `nedelen@nd.edu`. You may use Lee's and do Carmo's books, the class notes, and previous homeworks and midterms from this class, but no other resources. You are allowed to quote theorems/lemmas/corollaries from the book or class or previous homework *that we or the book have proven.* You are not allowed to quote statements from the books that are given without proof (e.g. exercises).

Q1: A. Let (M^n, g) be a complete Riemannian manifold ($n \geq 2$), and suppose M admits a geodesic line, i.e. there is a geodesic $\gamma : \mathbb{R} \to M$ parameterized by arclength which is minimizing on any finite segment. By considering variations of the form $\phi(t)e_i(t)$, where $\{e_i(t)\}_i$ is an ON parallel frame along γ, show that the following inequality must be true:

$$\int_{\mathbb{R}} (n-1)\phi'(t)^2 - \mathrm{Ric}|_{\gamma(t)}(\gamma', \gamma')\phi(t)^2 dt \geq 0 \quad \forall \phi \in C^1_c(\mathbb{R}).$$

B. Using part A, deduce that M cannot have positive Ricci curvature.

C. Use parts A, B to prove that any complete, non-compact (M^n, g) with positive Ricci curvature can have only one end. (Hint: use a result from the Midterm).

C. Prove that that $S^1 \times \mathbb{R}$ admits no complete metric of positive sectional curvature.

D. On the other hand, show that $\mathbb{R}^2$ does admit a complete metric of positive sectional curvature. (Hint: use a result from Homework 9).

Q2: Let (M^n, g) be a complete, connected Riemannian manifold, and $f : M \to \mathbb{R}$ a smooth function. Write ∇f for the gradient of f.

A. Suppose that $\nabla f \neq 0$ for all $x \in f^{-1}(0)$. Show that $S := f^{-1}(0)$ is a smooth embedded hypersurface, $\frac{\nabla f}{|\nabla f|}$ is a choice of unit normal for S, and the second fundamental form of S (with the induced metric) can be expressed as

$$B(X, Y) = -\frac{\nabla^2 f(X, Y)}{|\nabla f|^2} \nabla f.$$

B. Suppose that $|\nabla f| \equiv 1$. Prove that the integral curves of f are geodesics. Deduce that the flow $\phi_t(x)$ of ∇f exists for all $t \in \mathbb{R}$. Bonus: show the integral curves are minimizing geodesics.

C. Suppose that $\nabla^2 f \equiv 0$, and f is not constant. Prove that M isometrically splits off a line, in the sense that there is an $(\hat{M}^{n-1}, \hat{g})$ so that (M, g) is isometric to $(\hat{M} \times \mathbb{R}, \hat{g} + dr)$. Hint: Use a result from the Midterm.

Q3: Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds, and let $(M_1 \times M_2, g_1 + g_2)$ be the product manifold endowed with the product metric. For $p = (p_1, p_2) \in M_1 \times M_2$, $X \in T_{(p_1,p_2)}(M_1 \times M_2)$, we can write $X = X_1 + X_2$ (orthogonal direct sum) for $X_i \in T_{p_i} M_i$.

A. Given vectors $X, Y \in T_p(M_1 \times M_2)$, show that the sectional curvature K of the plane spanned by X, Y can be expressed as

$$K_{M_1 \times M_2}(p, X, Y) = \frac{|X_1 \wedge Y_1|^2 K_{M_1}(p_1, X_1, Y_1) + |X_2 \wedge Y_2|^2 K_{M_2}(p_2, X_2, Y_2)}{|X \wedge Y|^2},$$

where $|u \wedge v|^2 = |u|^2 |v|^2 - < u, v >^2$, and K_{M_i} are the sectional curvature functions on M_i.

B. Deduce that at every point there exist a plane (or planes) for which the sectional curvature of $g_1 + g_2$ is zero, irrespective of the sectional curvatures of M_1, M_2.

Q4: Is it possible (prove or disprove) to find a complete *non-planar* surface $S \subset \mathbb{R}^3$ so that:

A. For every $p \in S$, there is a line in $\mathbb{R}^3$ passing through p contained in S?

B. For every $p \in S$, there are two lines in $\mathbb{R}^3$ passing through p contained in S?

C. For every $p \in S$, there are three lines in $\mathbb{R}^3$ passing through p contained in S?