Q3.2.2 given $x \in I$

can find $\{q_n\} \subset \mathbb{Q}$ so that $q_n \to x$ (since $\mathbb{Q}$ dense)

since I open, $q_n \in I$ for n large

$\Rightarrow f(x) = \lim_n f(q_n) = 0$

Q3.2.3 define $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n} \end{cases}$ $(n \in \mathbb{N})$

Q3.2.4 let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for n odd

since $p(x) = 0 \iff \frac{p(x)}{a_n} = 0$

$\hookrightarrow$ enough to consider $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$

now for $x \gg 1$, $p(x) = x^n\left(1 + \frac{b_{n-1}}{x} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}\right)$

$$\geq x^n\left(1 - \frac{n \max_k |b_k|}{x}\right)$$

$$\geq \frac{x^n}{2} \quad \text{if} \quad x > 2n \max_k |b_k|$$

$$> 0$$

Similarly, if $x \ll -1$, $p(x) \le x^n\left(1 - \frac{n \max |b_k|}{|x|}\right)$

(since n odd) $\qquad \le \frac{x^n}{2} < 0$ if $x < -2n \max_k |b_k|$

so p takes on both positive and negative values

$\rightarrow$ IVT says $\exists\ x^*$ so that $p(x^*) = 0$

Q 3.2.5 consider $g(x) = f(x) - x$

since $f: [a,b] \to [a,b] \Rightarrow f(a) \geq a$
and $f(b) \leq b$

$\Rightarrow g(a) \geq a - a = 0$

$g(b) \leq b - b = 0$

and g continuous since f continuous

$\Rightarrow$ (IVT) $\exists c \in [a,b]$ s.t. $g(c) = 0$

$\Rightarrow f(c) = c$

Q3.3.1 spse $|f(x) - f(y)| \leq M|x-y| \quad \forall x,y \in S$

given $\varepsilon > 0$, choose $\delta = \varepsilon/M$

$\Rightarrow |f(x) - f(y)| \leq \varepsilon$ when $|x-y| \leq \delta$

Q3.3.2 if $x, y \geq a > 0 \Rightarrow |f(x) - f(y)|$

$$= \left|\frac{1}{x} - \frac{1}{y}\right|$$

$$\leq \frac{|x-y|}{xy}$$

$$\leq \frac{|x-y|}{a^2}$$

$$\leq \varepsilon \quad \text{if} \quad |x-y| \leq a^2 \varepsilon$$

$\Rightarrow f(x)$ unif. cont. on $[a, \infty)$

OTOH, consider $x_n = \frac{1}{n+1}$, $y_n = \frac{1}{n} \in (0, \infty)$

then $|x_n - y_n| \to 0$

but $|f(x_n) - f(y_n)| = |n+1-n| = 1 \not\to 0$

$\Rightarrow f$ not unif. continuous on $(0, \infty)$

Q3.3.3 I claim that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$

to see this observe that $a + b \leq a + b + 2\sqrt{a}\sqrt{b}$

$$= (\sqrt{a} + \sqrt{b})^2$$

$$\Rightarrow \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$$

now $\sqrt{x} \leq \sqrt{|x-y| + y} \leq \sqrt{|x-y|} + \sqrt{y}$ (for $x, y \geq 0$)

and similarly $\sqrt{y} \leq \sqrt{|x-y|} + \sqrt{x}$

$\Rightarrow |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$ $\forall x, y \geq 0$

so given $\varepsilon > 0$, chose $\delta = \varepsilon^2$ $\Rightarrow |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|} \leq \varepsilon$

for $|x-y| \leq \delta = \varepsilon^2$

Q3.3.4 given ε, choose δ so that:

$|f(x) - f(y)| \leq \varepsilon/2$ for $|x-y| \leq \delta$ and $x, y \in (a, c]$

and $|f(x) - f(y)| \leq \varepsilon/2$ for $|x-y| \leq \delta$ and $x, y \in [c, b)$

(which we can do by hypothesis)

now for $x, y \in (a, b)$ with $|x-y| \leq \delta$ and (WLOG) $x < y$

if $x, y \in (a, c]$ or $x, y \in [c, b)$ then $|f(x) - f(y)| \leq \varepsilon$

if $x \leq c \leq y$ then $|x - c| \leq \delta$ and $|c - y| \leq \delta$

$\Rightarrow |f(x) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$