Q 2.3.1

(A) $a_n = 3 + (-1)^n \Rightarrow \overline{a}_n = 4$ and $\underline{a}_n = 2 \quad \forall n$

$\Rightarrow \limsup a_n = \lim \overline{a}_n = 4$

$\liminf a_n = \lim \underline{a}_n = 2$

(B) $a_n = 3 + (-2)^n \Rightarrow \overline{a}_n = 5$ and $\underline{a}_n = 1 \quad \forall n$

$\Rightarrow \limsup a_n = 5$

$\liminf a_n = 1$

(C) $a_n = 3 + \frac{1}{n}\sin(n) \Rightarrow 3 - \frac{1}{n} \leq \underline{a}_n \leq \overline{a}_n \leq 3 + \frac{1}{n}$

$\Rightarrow 3 \leq \liminf a_n \leq \limsup a_n \leq 3$

$\Rightarrow 3 = \liminf a_n = \limsup a_n$

Q 2.3.2 if $\overline{a}_n = \infty \Rightarrow \liminf a_n \leq \infty = \limsup a_n$

if $\underline{a}_n = -\infty \Rightarrow \liminf a_n = -\infty \leq \limsup a_n$

if $\overline{a}_n < \infty$ and $\underline{a}_n > \infty$

then $\underline{a}_n \leq \overline{a}_n \quad \forall n$

$\Rightarrow \liminf a_n = \lim \underline{a}_n \leq \lim \overline{a}_n \leq \limsup a_n$

Q2.3.3

(A) we have $\overline{\alpha a_n} = \sup\{\alpha a_k : k \geq n\}$ $\quad(\alpha > 0)$

$$= \alpha \sup\{a_k : k \geq n\}$$

$$= \alpha \overline{a_n}$$

$$\Rightarrow \limsup \alpha a_n = \lim \overline{\alpha a_n} = \lim \alpha \overline{a_n}$$

$$= \alpha \lim \overline{a_n}$$

$$= \alpha \limsup a_n$$

likewise $\underline{\alpha a_n} = \alpha \underline{a_n}$ so $\liminf \alpha a_n = \alpha \liminf a_n$

(B) If $\alpha < 0$ then instead we get

$$\overline{\alpha a_n} = \sup\{\alpha a_k : k \geq n\}$$

$$= \alpha \inf\{a_k : k \geq n\}$$

$$= \alpha \underline{a_n}$$

$$\Rightarrow \limsup \alpha a_n = \lim \overline{\alpha a_n} = \lim \alpha \underline{a_n}$$

$$= \alpha \lim \underline{a_n} = \alpha \liminf a_n$$

and likewise $\underline{\alpha a_n} = \alpha \overline{a_n}$

$$\Rightarrow \liminf \alpha a_n = \alpha \limsup a_n$$

Q 2.3.4

since $a_n \le b_n \ \forall n$

$\Rightarrow a_k \le \sup\{b_k : k \geq n\} = \overline{b}_n \quad \forall k \geq n$

$\Rightarrow \overline{a}_n \le \overline{b}_n$

$\Rightarrow \limsup a_n = \lim \overline{a}_n \le \lim \overline{b}_n = \limsup b_n$

and likewise $\underline{a_n} \le \underline{b_n} \Rightarrow \liminf a_n \le \liminf b_n$

Q3.1.1 we have $\big||f(x)| - |\ell|\big| \le |f(x) - \ell|$ by Δ-inequality

so $\big||f(x)| - |\ell|\big| \le |f(x) - \ell| \le \varepsilon$

provided $|x - c| \le \delta$

(by defn of limit $f(x) \to \ell$)

Q3.1.2 observe $|f(x) - 1| < \varepsilon$

$\iff |2x - 1 - 1| < \varepsilon$

$\iff |2x - 2| < \varepsilon$

$\iff 2|x-1| < \varepsilon$

$\iff |x - 1| < \frac{\varepsilon}{2}$

so optimal $\delta = \frac{\varepsilon}{2}$ (optimal meaning "largest" here)

Q3.1.3 (A) $|x^2 - 1| = |x+1|\,|x-1| \le (|x|+1)|x-1|$

$\le 3|x-1|$ provided $|x-1| \le 1$

($\Rightarrow |x| \le |x-1| + 1 = 2$)

$\le \varepsilon$

provided $|x-1| \le \varepsilon/3$

so take $\delta = \min(1, \frac{\varepsilon}{3})$

(B) $|x^2 - c^2| = |x+c|\,|x-c| \le (|x| + c)\,|x-c|$

$\le (2c+1)|x-c|$ provided $|x-c| \le 1$

($c > 0$) ($\Rightarrow |x| \le 1 + c$)

$\le \varepsilon$ provided $|x-c| \le \frac{\varepsilon}{2c+1}$

so take $\delta = \min(1, \frac{\varepsilon}{2c+1})$

Q3.1.4 (A) for $c > 0$

$$|\sqrt{x}-\sqrt{c}| = \left|(\sqrt{x}-\sqrt{c})\frac{(\sqrt{x}+\sqrt{c})}{\sqrt{x}+\sqrt{c}}\right|$$

$$= \frac{|x-c|}{\sqrt{x}+\sqrt{c}}$$

$$\leq \frac{|x-c|}{\sqrt{c}} \leq \varepsilon \quad \text{provided} \quad |x-c| \leq \underbrace{\sqrt{c}\,\varepsilon}_{=\delta}$$

(B) $|\sqrt{x}-0| = \sqrt{x} \leq \varepsilon$ provided $|x| \leq \underbrace{\varepsilon^2}_{=\delta}$