Q2.2.5 write $I_n = [a_n, b_n]$

since $[a_1, b_1] \supset [a_2, b_2] \supset \dots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}]$

we have $a_1 \le a_n \le a_{n+1} \le b_{n+1} \le b_n \le b_1$

$\Rightarrow$ $\{a_n\}$ increasing + bdd, $\{b_n\}$ decreasing + bdd

$\Rightarrow$ $a_n \to a$ and $b_n \to b$

and $a \le b$ since $a_n \le b_n$

take $p = a$ $\Rightarrow$ $\forall n$, $a_n \le a_{n+1} \le \overset{p}{\overset{\shortparallel}{a}} \le b \le b_{n+1} \le b_n$

$\Rightarrow$ $p \in [a_n, b_n]$ $\forall n$

Q2.2.6 set $a_0 = a$, $b_0 = b$ $(a, b > 0)$

$\hookrightarrow$ $a_{n+1} = \dfrac{a_n + b_n}{2}$ $b_{n+1} = \sqrt{a_n b_n}$ $\forall n \ge 0$

notice formulas are symmetric in a_n, b_n

and nothing is changed

$\Rightarrow$ if $a < b$ can swap a and b

$\Rightarrow$ WLOG assume either $a > b$ or $a = b$

"without loss of generality"

case $a > b$: we first observe that $a_n \ge b_n$ for all n

$\hookrightarrow$ true for $n = 0$ by assumption

for general n, we have $a_{n+1} = \dfrac{a_n + b_n}{2}$

$\ge \sqrt{a_n b_n}$ by AM-GM inequality

$= b_{n+1}$

(it is clear from construction that every $a_n > 0$, $b_n > 0$)

now we compute $a_{n+1} - a_n = \frac{a_n + b_n}{2} - a_n$

$$= \frac{b_n - a_n}{2} \le 0 \quad \text{since } a_n \ge b_n$$

and $b_{n+1} - b_n = \sqrt{a_n b_n} - b_n$

$$\ge \sqrt{b_n \cdot b_n} - b_n \quad \text{since } a_n \ge b_n$$

$$= 0$$

so $\{a_n\}$ is decreasing, $\{b_n\}$ increasing,

and $a_1 \ge a_n \ge a_{n+1} \ge b_{n+1} \ge b_n \ge b_1$

$\Rightarrow \{a_n\}, \{b_n\}$ both bounded

$\Rightarrow a_n \to A$ and $b_n \to B$

take limit of equation $a_{n+1} = \frac{a_n + b_n}{2}$

$$\downarrow \qquad \downarrow$$

$$A = \frac{A+B}{2}$$

$$\Rightarrow 0 = \frac{A-B}{2} \Rightarrow A = B$$

so in fact $a_n \to l$ and $b_n \to l$ for some $l \in \mathbb{R}$

case $a = b$: if $a = b$ then $a_{n+1} = a$ and $b_{n+1} = b = a \ \forall n$

$\Rightarrow$ statement becomes trivial

Q2.2.7 as before let $a_0 = a$, $b_0 = b$ $(a, b > 0)$

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{2a_nb_n}{a_n+b_n} \quad (n \geq 0)$$

and since formulas symmetric, WLOG can assume $a > b$ or $a = b$

case $a > b$: we first claim that $a_n \geq b_n \ \forall n$

↳ true for $n = 0$ by assumption

↳ $$a_{n+1} - b_{n+1} = \frac{a_n+b_n}{2} - \frac{2a_nb_n}{a_n+b_n}$$

$$= \frac{(a_n+b_n)^2 - 4a_nb_n}{a_n+b_n}$$

$$= \frac{a_n^2 + 2a_nb_n + b_n^2 - 4a_nb_n}{a_n+b_n}$$

$$= \frac{(a_n - b_n)^2}{a_n+b_n} \geq 0$$

(clearly $a_n > 0$ and $b_n > 0$ by construction)

now compute: $a_{n+1} - a_n = \frac{a_n+b_n}{2} - a_n = \frac{b_n - a_n}{2} \leq 0$ since $a_n \geq b_n$

$$b_{n+1} - b_n = \frac{2a_nb_n}{a_n+b_n} - b_n$$

$$= \frac{2a_nb_n - a_nb_n - b_n^2}{a_n+b_n}$$

$$= \frac{b_n(a_n - b_n)}{a_n+b_n} \geq 0 \quad \text{since } a_n \geq b_n$$

so $\{a_n\}$ decreasing, $\{b_n\}$ increasing

and $a_0 \geq a_n \geq a_{n+1} \geq b_{n+1} \geq b_n \geq b_0 \implies \{a_n\}, \{b_n\}$ bounded

$\Rightarrow a_n \to A$ and $b_n \to B$ for some $A, B \in \mathbb{R}$

take limit of
$$\begin{array}{ccc} a_{n+1} & = & \frac{a_n + b_n}{2} \\ \downarrow & & \downarrow \\ A & = & \frac{A+B}{2} \end{array} \Rightarrow A = B$$

So in fact $a_n \to \ell$ and $b_n \to \ell$ for some $\ell \in \mathbb{R}$

Finally, observe that
$$a_{n+1} b_{n+1} = \frac{a_n + b_n}{2} \cdot \frac{2 a_n b_n}{a_n + b_n}$$
$$= a_n b_n$$
$$= \quad = a_0 b_0 = ab \quad \forall n$$

$$\Rightarrow \lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} ab$$

$$\Rightarrow \ell^2 = ab \Rightarrow \ell = \sqrt{ab}$$

Q2.2.4 $\quad a_n \le b_n$

$\Rightarrow \quad \bar{a}_n = \sup\{a_k : k \ge n\} \le \sup\{b_k : k \ge n\} = \bar{b}_n$

and $\quad \underline{a}_n = \inf\{a_k : k \ge n\} \le \inf\{b_k : k \ge n\} = \underline{b}_n$

So $\quad \limsup a_n = \lim \bar{a}_n \le \lim \bar{b}_n = \limsup b_n$

$\liminf a_n = \lim \underline{a}_n \le \lim \underline{b}_n = \liminf b_n$

Q2.4.2

(a) $\sup \underbrace{\{\tfrac{1}{n} : n \in \mathbb{N}\}}_{S} = 1 \quad$ since $\quad 1 \ge \frac{1}{n} \ \forall n \in \mathbb{N}$ and $1 = \frac{1}{1} \in S$

$\inf S = 0 \quad$ since $\quad 0 \le \frac{1}{n} \ \forall n \in \mathbb{N}$ and $\frac{1}{n} \xrightarrow[n \to \infty]{} 0$

(b) $\sup\{2^n : n \in \ \} = \infty \quad$ since $\quad 2^n \to \infty$ as $n \to \infty$

$\inf\{2^n : n \in \mathbb{Z}\} = 0 \quad$ since $\quad 2^n > 0 \ \forall n$ and $2^n \xrightarrow[n \to -\infty]{} 0$

(c) $\sup\{x^2 : -1 < x < 1\} = 1 \quad$ since $\quad 1 \ge x^2 \ \forall x \in (-1, 1)$
and $x^2 \to 1$ as $x \to \pm 1$

$\inf \underbrace{\{x^2 : -1 < x < 1\}}_{S} = 0 \quad$ since $\quad x^2 \ge 0 \ \forall x$ and $0 = 0^2 \in S$

Q2.4.2

(A) $\sup\{1/n : n \in \mathbb{N}\} = 1$ since $1 \geq \frac{1}{n}\ \forall n \in \mathbb{N}$

and $1 = \frac{1}{1}$ lies in set

$\inf\{\text{———}\} = 0$ since $0 < \frac{1}{n}\ \forall n \in \mathbb{N}$

and $\frac{1}{n} \to 0$ as $n \to \infty$

(B) $\sup\{2^n, n \in \mathbb{Z}\} = \infty$ since $2^n \to \infty$ as $n \to \infty$

$\inf\{\text{———}\} = 0$ since $2^n > 0\ \forall n \in \mathbb{Z}$ and $2^n \to 0$ as $n \to -\infty$

(C) $\sup\{x^2 ; -1 < x < 1\} = 1$ since $1 \geq x^2\ \forall x \in (-1,1)$

and $x^2 \to 1$ as $x \to \pm 1$

$\inf\{\text{———}\} = 0$ since $x^2 \geq 0\ \forall x$ and $0 = 0^2$ lies in se[t]

Q2.4.7

If $\alpha > 0$, observe $\alpha \sup S \geq \alpha x\ \forall x \in S \Rightarrow \alpha \sup S$ is an upper bound for αS

$\Rightarrow \alpha \sup S \geq \sup(\alpha S)$

On the other hand, $\sup(\alpha S)$ is an upper bound for αS

$\Rightarrow \sup(\alpha S) \geq \alpha x\ \forall x \in S$

$\Rightarrow \frac{1}{\alpha}\sup(\alpha S) \geq x\ \forall x \in S$

$\Rightarrow \frac{1}{\alpha}\sup(\alpha S) \geq \sup(S) \Rightarrow \sup(\alpha S) \geq \alpha \sup(S)$

So we must have $\sup(\alpha S) = \alpha \sup(S)$

we claim $\sup(-S) = -\inf S$

well, $\sup S \geq x \;\; \forall x \in S \implies -\sup(S) \leq -x \;\; \forall x \in S$

$\implies -\sup(S) \leq \inf(-S)$

OTOH, $\inf(-S) \leq -x \;\; \forall x \in S$
"on the other hand"

$\implies -\inf(-S) \geq x \;\; \forall x \in S$

$\implies -\inf(-S) \geq \sup S$

$\implies \inf(-S) \leq -\sup(S)$

so we must have $\inf(-S) = -\sup(S)$

now if $\alpha > 0$ then $\inf(\alpha S) = -\sup(-\alpha S)$

$= -\sup(\alpha(-S))$

$= -\alpha \sup(-S)$

$= \alpha \inf(S)$

if $\alpha < 0$ then $\sup(\alpha S) = \sup((-\alpha)(-S))$

$= (-\alpha)\sup(-S)$

$= (-\alpha)(-\inf S)$

$= \alpha \inf(S)$

and $\inf(\alpha S) = \inf((-\alpha)(-S)) = (-\alpha)\inf(-S)$

$= (-\alpha)(-\sup S) = \alpha \sup S$