Q 7.1.5   Spse $a_n \to a$

$\hookrightarrow |a_n^2 - a^2| \le |a_n - a|\,|a_n + a|$ since $(c-d)(c+d) = c^2 - d^2$

given $\varepsilon > 0$

$\le |a_n - a|\,(|a_n| + |a|)$

$\le |a_n - a|\,(1 + 2|a|) \le \varepsilon$

provided $n \geq N$ chosen so that

$|a_n - a| \le \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$

so $a_n^2 \to a$

conversely, if $a_n = (-1)^n$ then $a_n^2 = 1 \to 1$

but a_n does not converge

Q 7.1.8 $a_n \to \infty$

$\Leftrightarrow \forall M > 0\ \exists N \in \mathbb{N}$ s.t. $a_n \geq M$ for $n \geq N$

$\Leftrightarrow \forall M > 0\ \exists N \in \mathbb{N}$ st $\frac{1}{a_n} \le \frac{1}{M}$ for $n \geq N$

$\Leftrightarrow \forall \varepsilon = \frac{1}{M}\ \exists N \in \mathbb{N}$ st $0 < \frac{1}{a_n} \le \varepsilon$ for $n \geq N$

since $a_n > 0$

$\Leftrightarrow \frac{1}{a_n} \to 0$

Q 7.1.9 (a) let $a_n = \alpha/n$, $b_n = 1/n$

$\Rightarrow \frac{a_n}{b_n} = \alpha \to \alpha$ and $a_n \to 0$, $b_n \to 0$

(b) let $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$

$\Rightarrow \frac{a_n}{b_n} = \frac{1/n}{1/n^2} = n \to \infty$ and $a_n \to 0$, $b_n \to 0$

(c) let $a_n = -\frac{1}{n}$, $b_n = \frac{1}{n^2}$ (both $\to 0$)

$\Rightarrow \frac{a_n}{b_n} = -n \to -\infty$

(d) let $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{1}{n}$ (both $\to 0$)

$\Rightarrow \frac{a_n}{b_n} = (-1)^n$ does not converge

Q 7.1.10 (a) let $a_n = \alpha + n$, $b_n = n$ (both $\to \infty$)

$\Rightarrow a_n - b_n = \alpha \to \alpha$

(b) let $a_n = n^2$, $b_n = n$ (both $\to \infty$)

$$\begin{aligned}\Rightarrow a_n - b_n &= n^2 - n \\ &= n(n-1) \\ &\geq n \quad (\text{for } n \geq 2) \\ &\to \infty\end{aligned}$$

(c) let $a_n = n$, $b_n = n^2$

$\Rightarrow a_n - b_n = n - n^2 \to -\infty$ (by same reasoning)

(d) let $a_n = (-1)^n + n$, $b_n = n$

$\Rightarrow a_n - b_n = (-1)^n$ does not converge now $\lim_n a_{n+1} = \lim_n \frac{a_n^2+c}{2a_n}$

$\Rightarrow \quad l = \frac{l^2+c}{2l} \Rightarrow 2l^2 = l^2+c$

$\Rightarrow l = \sqrt{c}$

Q 2.2.4 let $a_n = nc^n$ for $0 < c < 1$

observe $a_{n+1} = (n+1)c^{n+1}$

$= \left(\frac{n+1}{n} c\right) nc^n$

$= \left(\left(\frac{n+1}{n}\right) c\right) a_n \leq a_n$

provided $\frac{n+1}{n} c \leq 1$

so, provided $(n+1)c \leq n$

so, provided $\frac{c}{1-c} \leq n$

so $\{a_n\}$ decreasing for $n \geq \frac{c}{1-c}$

and trivially $a_n \geq 0 \Rightarrow a_n$ converges to some limit l

now $\lim_n a_{n+1} = \lim_n \left(\frac{n+1}{n} c\right) a_n$

$\downarrow$

$1 \cdot c$

$\Rightarrow \quad l = cl$

$\Rightarrow l(1-c) = 0$

$\Rightarrow l = 0$ since $c \neq 1$

Q. 7.2.1. given: $\{a_n\}_n$ monotone, and some subseq $a_{n_k} \xrightarrow[k\to\infty]{} a$

want to show: $a_n \to a$ also

assume a_n increasing

know that $\forall \varepsilon\ \exists K$ so that $|a_{n_k} - a| \leq \varepsilon$ for $k \geq K$

$\Rightarrow \quad a \geq a_{n_k} \geq a - \varepsilon$ for $k \geq K$

since a_{n_k} increasing also

given $n \geq n_K$, can find $n_{k'} \geq n$

$\Rightarrow \quad a \geq a_{n_{k'}} \geq a_n \geq a_{n_K} \geq a - \varepsilon$

$\Rightarrow \quad a \geq a_n \geq a - \varepsilon \quad \forall\ n \geq n_K$

if a_n decreasing apply above reasoning to $-a_n$ (which is increasing)

$\Rightarrow \quad -a_n \to -a$

$\Rightarrow \quad a_n \to a$

Q. 7.7.2 $\quad a_1 = \sqrt{c}$ for $c > 0$

$a_{n+1} = \sqrt{c + a_n}$

being the positive square root

clearly every $a_n \geq 0$

if there is a finite limit l

↳ then $\lim_n a_{n+1} = \lim_n \sqrt{c + a_n} \quad \Rightarrow \quad l = \sqrt{c + l}$

$\Rightarrow \quad l^2 = c + l$

$\Rightarrow \quad l^2 - l - c = 0$

$\Rightarrow \quad l = \dfrac{1 + \sqrt{1 + 4c}}{2}$

(since $l \geq 0$)

now observe that $a_{n+1} - a_n$

$$= \sqrt{c+a_n} - a_n$$

$$= \frac{(\sqrt{c+a_n} - a_n)(\sqrt{c+a_n} + a_n)}{(\sqrt{c+a_n} + a_n)}$$

$$= \frac{c+a_n - a_n^2}{\sqrt{c+a_n} + a_n}$$

$$\geq 0 \iff c + a_n - a_n^2 \geq 0$$

$$\iff a_n^2 - a_n - c \leq 0$$

$$\iff a_n \leq l$$

so let us try to show $a_n \leq l \quad \forall n$

$a_1 = \sqrt{c} = \frac{\sqrt{4c}}{2} \leq \frac{1+\sqrt{1+4c}}{2} = l$ so true for $n=1$

if $a_n \leq l$

then $a_{n+1} = \sqrt{c+a_n} \leq \sqrt{c+l} = l$ by definition of l

so true for $n+1$ if true for n

$\Rightarrow a_n \leq l \quad \forall n$ by induction

so $a_n \leq l \quad \forall n \Rightarrow a_{n+1} - a_n \geq 0$

$\Rightarrow a_n$ increasing and bounded above by l

$\Rightarrow a_n$ has a (finite) limit

$\Rightarrow a_n \to l$ since limit must be l

Q 7.7.3 define $a_1 = c$ for $c > 0$

$$a_{n+1} = \frac{a_n^2 + c}{2a_n}$$

Claim: $a_n \geq \sqrt{c}$ for $n \geq 2$

$$a_2 = \frac{a_1^2 + c}{2a_1} = \frac{c^2 + c}{2c} = \frac{c+1}{2} \geq \sqrt{c \cdot 1} = \sqrt{c}$$ by AM-GM inequality

and $$a_{n+1} = \frac{a_n^2 + c}{2a_n}$$

$$= \frac{a_n + \frac{c}{a_n}}{2} \geq \sqrt{a_n \cdot \frac{c}{a_n}} = \sqrt{c}$$ for $n \geq 1$

(clearly $a_n > 0$ for all n)

now observe that $a_{n+1} - a_n$

(if $n \geq 2$)

$$= \frac{a_n^2 + c}{2a_n} - a_n$$

$$= \frac{a_n^2 + c - 2a_n^2}{2a_n}$$

$$= \frac{c - a_n^2}{2a_n}$$

$$\leq \frac{c - (\sqrt{c})^2}{2a_n}$$ since $a_n \geq \sqrt{c}$

$$\leq 0$$

so $\{a_n\}$ decreasing for $n \geq 2$ and bounded below by $\sqrt{c}$ $\Rightarrow$ a_n converges to some limit l