Q 1.2.1 by $\triangle$-ineq we have $|x| = |x-y+y|$

$$\leq |x-y| + |y|$$

and $|y| = |y-x+x|$

$$\leq |y-x| + |x|$$

$$= |x-y| + |x|$$

$\Rightarrow -|x-y| \leq |x|-|y| \leq |x-y|$

$\Rightarrow \big|\, |x|-|y| \,\big| \leq |x-y|$

Q.1.2.2 $(a-b)^2 \geq 0$

$\Rightarrow a^2 - 2ab + b^2 \geq 0$

$\Rightarrow a^2 + b^2 \geq 2ab$

Q 1.2.3 observe that $\dfrac{a+c}{b+d} > \dfrac{a}{b}$

$\Leftrightarrow (a+c)b > (b+d)a$

$\Leftrightarrow cb > ad$ which is our assuption

similarly, $\dfrac{a+c}{b+d} < \dfrac{c}{d}$

$\Leftrightarrow (a+c)d < (b+d)c$

$\Leftrightarrow ad < bc$

Q 2.1.1 A.

observe: $\left|\frac{\sin n}{\sqrt{n}}\right| < \varepsilon \Leftarrow \frac{1}{\sqrt{n}} < \varepsilon$ (since $|\sin(n)| \le 1$)

$\iff \frac{1}{\varepsilon^2} < n$

therefore if $n \ge N > \frac{1}{\varepsilon^2}$ then $\left|\frac{\sin(n)}{\sqrt{n}} - 0\right| < \varepsilon$

B. note that $\frac{2^n}{n!} = \frac{2 \cdot 2 \cdots 2}{n(n-1) \cdots 2\,1}$

$\le \frac{2}{n}$

so $\left|\frac{2^n}{n!}\right| < \varepsilon \Leftarrow \frac{2}{n} < \varepsilon$

$\iff \frac{2}{\varepsilon} < n$

so if $n \ge N > \frac{2}{\varepsilon}$ then $\left|\frac{2^n}{n!} - 0\right| < \varepsilon$

C. we have: $\left|\sqrt{n+4} - \sqrt{n}\right| < \varepsilon$

$\iff \left|\frac{(\sqrt{n+4} - \sqrt{n})(\sqrt{n+4} + \sqrt{n})}{\sqrt{n+4} + \sqrt{n}}\right| < \varepsilon$

$\iff \left|\frac{n+4-n}{\sqrt{n+4} + \sqrt{n}}\right| < \varepsilon$

$\iff \frac{4}{\sqrt{n+4} + \sqrt{n}} < \varepsilon \Leftarrow \frac{2}{\sqrt{n}} < \varepsilon \iff \frac{4}{\varepsilon^2} < n$

so if $n \ge N > 4/\varepsilon^2$ then $\left|\sqrt{n+4} - \sqrt{n} - 0\right| < \varepsilon$

Q1.2.4 proof by induction that $1^2 + 2^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$

notice that $\frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2 \cdot 1 + 1)$

$$= \frac{2 \cdot 3}{6} = 1 = 1^2 \quad \Rightarrow \text{ statement true for } n=1$$

assume true for n

$$\Rightarrow 1^2 + 2^2 + \dots + n^2 + (n+1)^2$$

$$= \frac{1}{6} n(n+1)(2n+1) + (n+1)^2$$

$$= \frac{(n+1)}{6} \left[n(2n+1) + 6(n+1) \right]$$

$$= \frac{(n+1)}{6} \left[2n^2 + 7n + 6 \right]$$

$$= \frac{(n+1)}{6} (n+2)(2n+3) \quad \Rightarrow \text{ statement true for } n+1$$

by induction statement true $\forall n \in \mathbb{N}$

Q.2.1.2 if $a > 0$ then $|a| = a$,

and for $n \geq N \gg 1$ $a_n > a/2 > 0$

$\Rightarrow |a_n| = a_n$ also

so $||a_n| - |a|| = |a_n - a|$ for all n large

and $|a_n - a| \to 0 \Rightarrow ||a_n| - |a|| \to 0$

if $a < 0$ then apply same reasoning to $-a_n$ and $-a$.

if $a = 0$ then $||a_n| - 0| = |a_n| = |a_n - 0|$

$\downarrow$

0 by assumption

so $||a_n| - 0| \to 0$ also

Q 2.2.3 by assumption $|a_n| \leq M$ $\forall n \in \mathbb{N}$, for some $M > 0$

now $|a_n b_n - 0| = |a_n b_n|$

$\leq M |b_n|$

$\leq \varepsilon$ if $n \geq N$ large so that

$|b_n| \leq \varepsilon / M$

Q. 2.2.4 spse first $a > 0$

$\Rightarrow |\sqrt{a_n} - \sqrt{a}| = \left| (\sqrt{a_n} - \sqrt{a}) \left(\frac{\sqrt{a_n} + \sqrt{a}}{\sqrt{a_n} + \sqrt{a}} \right) \right|$

$= \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| \leq \frac{|a_n - a|}{\sqrt{a}} < \varepsilon$

if $n \geq N$ large so that

$|a_n - a| \leq \sqrt{a}\, \varepsilon$

spse $a = 0$

$\Rightarrow |\sqrt{a_n} - \sqrt{a}| = \sqrt{a_n} < \varepsilon$ if $n \geq N$ large so that

$|a_n| < \varepsilon^2$