Math 60670 Homework 9

Due Friday, April 30. Note you have two weeks to do this homework.

Problem 1: Let G be a Lie group, and write e for the identity element. Given an $a \in G$, the operators $L_a(x) = a \cdot x$ and $R_a(x) = x \cdot a$ are called left-/right-translation (resp.). A vector field X in G is left-invariant if $X(a) = DL_a|_e X(e)$, and right-invariant if $X(a) = DR_a|_e X(e)$. A metric g on G is called left- (resp. right-)invariant if

$$(L_a)^* g = g, \quad (\text{resp. } (R_a)^* g = g) \quad \forall a \in G.$$

The metric is called bi-invariant if it is both left- and right-invariant. The adjoint map $Ad_a : G \to G$ is defined to be $Ad_a := L_a \circ R_{a^{-1}} \equiv R_{a^{-1}} \circ L_a$. In this question we assume that G is a Lie group with a bi-invariant metric g.

A. Let X be a left-invariant vector field on G, and let ϕ_t be the flow of X. Show that $\phi_t(x) = R_{\phi_t(e)}(x)$, and deduce that ϕ_t exists for all t.

B. Prove that if X, Y are left-invariant vector fields, then

$$\mathcal{L}_X Y|_e = \frac{d}{dt}\Big|_{t=0} DAd_{\phi_t(e)}(Y),$$

where $\mathcal{L}_X Y|_p = \lim_{t\to 0} \frac{1}{t}(D\phi_{-t} Y|_{\phi_t(p)} - Y|_p) = [X, Y]$ is the Lie derivative as defined earlier in the course.

C. Use part B. to show that

$$< [X, Y], Z > + < Y, [X, Z] >= 0$$

for all left-invariant vector fields X, Y, Z.

D. Deduce that the Levi-Civita connection ∇ w.r.t. this metric can be expressed as

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

where again X, Y are left-invariant.

E. Using the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, show that the curvature endomorphism $R(X,Y)Z$ takes the form

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]],$$

and deduce that the sectional curvature of (G,g) for the plane spanned by X, Y is

$$K(p,X,Y) = \frac{1}{4}\frac{|[X,Y]|^2}{|X \wedge Y|^2}.$$

Notice this is always non-negative, and is independent of p.

F. Identify 3×3 matrices with $\mathbb{R}^9$ in the obvious way, and show that $(A,B) := tr(A^T B)$ coincides with the Euclidean inner product under this identication. Let (G,g) be $SO(3)$ with this inner product (recall that $SO(3)$ consists of 3×3 matrices A for which $A^T A = Id$). Show that g is bi-invariant, and that the Lie algebra $\mathfrak{g}$ of G (which is just the tangent space of G at the identity Id) is the space of anti-symmetric 3×3 matrices.

G. Pick an ON basis E_1, E_2, E_3 for $\mathfrak{g}$, and compute $[E_i, E_j]$ in terms of the E_i. Prove that $SO(3)$ has constant sectional curvature $1/8$. Does $SO(4)$ also have constant sectional curvature?

H. (Bonus) The previous problem and the classification of spaceforms implies that the universal cover of $SO(3)$ is S^3 (up to scaling). In fact S^3 is the double cover of $SO(3)$, and $SO(3)$ is diffeomorphic to 3 dimensional projective space. Construct a double-covering map $S^3 \to SO(3)$.