Math 60670 Homework 8

Due Friday, April 16.

Problem 1: Let (M^n, g) be a Riemannian manifold, and $p \in M$, and x^i normal coordinates of M about p.

A. Show that the metric g_{ij} in these normal coordinates admits the Taylor expansion about 0:

$$g_{ij}|_x = \delta_{ij} - \frac{1}{3} R_{iklj}|_p x^k x^l + O(|x|^3). \quad (1)$$

Hint: Let $\gamma(t) = tv$ be a radial geodesic, and $J(t) = tW^i \partial_i$ be a Jacobi field along γ, and compute the first four t-derivatives of $|J(t)|^2$ at 0 in two ways.

B. Use this to show that the volume form has the expansion

$$dV|_{tv} = 1 - (t^2/6)\mathrm{Ric}|_p(v,v) + O(t^3)$$

and thereby deduce that the volume of a small geodesic ball $B_r(p) \subset M$ has the expansion

$$\mathrm{Vol}_g(B_r(p)) = \omega_n r^n \left(1 - \frac{r^2 \mathrm{Scal}(p)}{6(n+2)} + O(r^3)\right), \quad (2)$$

where ω_n is the Euclidean volume of the Euclidean unit ball. Hint: use HW5 Q5.

C. Bonus: Compute one further term in expansion (1) to get that the error in (2) is in fact $O(r^4)$.

Problem 2: Let $\bar{g} = e^{2u} g$ be a conformal metric on some M^n.

A. If ∇ is the Levi-Civita connection w.r.t. g, show that the Levi-Civita connection $\bar{\nabla}$ w.r.t. $\bar{g}$ takes the form

$$\bar{\nabla}_X Y = \nabla_X Y + X(u)Y + Y(u)X - g(X,Y)\mathrm{grad}(u),$$

where $\mathrm{grad}(u) = g^{ij}(\partial_i u)\partial_j$ is the gradient of u w.r.t. g.

B. If $n = 2$, show that the Gauss curvature $\bar{K}$ of $\bar{g}$ can be written

$$e^{2u}\bar{K} = -\Delta u + K,$$

where $\Delta u = \mathrm{tr}(\Delta^2 u)$ is the connection Laplacian of u w.r.t. g.

C. Use this to show that if $\alpha \in \mathbb{R}$, then the metric $g = \frac{g_{\mathrm{eucl}}}{(1+\alpha|x|^2)^2}$ defined on $\mathbb{R}^2$ (if $\alpha \geq 0$) or on $\{x : |x| < 1/|\alpha|\}$ (if $\alpha < 0$) has Gauss curvature $= 4\alpha$.