

Math 60670 Homework 7

Due Friday, April 9.

Problem 1: A Riemannian submanifold $M \subset (\bar{M}, \bar{g})$ (with the induced metric $g = \bar{g}|_{TM}$), is called totally geodesic if for every $p \in M$, $v \in T_pM$, the \bar{g} -geodesic γ with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$ lies in M . Show the following are equivalent:

- A. M is totally geodesic,
- B. Every g -geodesic in M is also a \bar{g} -geodesic in \bar{M} ,
- C. The second fundamental form of M vanishes.

Problem 2: Let $M \subset \mathbb{R}^3$ be the catenoid, which is the surface of revolution obtained by revolving the curve $x = \cosh z$ around the z -axis. Show that M has zero mean curvature.

Problem 3: A geodesic triangle in a Riemannian 2-manifold (M^2, g) is a domain Ω with piecewise-smooth boundary $\partial\Omega$ consisting of three geodesics meeting at three vertices. If M has constant Gauss curvature K , show that the sum of interior angles of any geodesic triangle is $\pi + KA$, where A is the area of Ω .

Problem 4: Let $M \subset (\bar{M}, \bar{g})$ be a compact, embedded Riemannian submanifold, and let NM be the normal bundle, i.e. the subbundle of $T\bar{M}$ consisting of vectors $v \in T_x\bar{M}$ for $x \in M$, $v \perp T_xM$. Let $(NM)_\epsilon = \{V \in NM : |V| < \epsilon\}$, and let $M_\epsilon = \{x \in \bar{M} : \text{dist}(x, M) < \epsilon\}$.

A. Show that for all small $\epsilon > 0$, the exponential map restricted to $(NM)_\epsilon$ is a diffeomorphism from $(NM)_\epsilon$ to M_ϵ . For these ϵ , the open set M_ϵ is called a tubular neighborhood of M .

B. If $r(x)$ denotes the distance from $x \in \bar{M}$ to M , show that r^2 is a smooth function on any tubular neighborhood M_ϵ . Give an example where r^2 is not smooth on all of \bar{M} .