Math 60670 Homework 6

Due Friday, March 19.

Problem 1: A. The Laplacian of a function f is defined to be the trace of the Hessian: $\Delta f = tr_g(\nabla^2 f)$. Show that one can alternatively write

$$\Delta f = \mathrm{div}(\mathrm{grad} f),$$

and hence in coordinates

$$\Delta f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j f), \quad g = \det g_{ij}.$$

B. A function $u \in C^2(M)$ is called harmonic if $\Delta u = 0$. If (M, g) is compact with no boundary, prove that any harmonic function is necessarily constant.

Problem 2: Let ω be a 1-form on (M, g). Show that

$$(R(X,Y)\omega)(Z) := (\nabla^2_{X,Y}\omega)(Z) - (\nabla^2_{Y,X}\omega)(Z) = -\omega(R(X,Y)Z).$$

Problem 3: Let ∇ be a linear connection on M (not necessarily the Levi-Civita connection). Let $\{E_i\}_i$ be a local frame on some open subset $U \subset M$, and let $\{\phi^i\}_i$ be the dual frame.

A. Show there is a uniquely determined matrix of 1-forms ${\omega_i}^j$ on U, called the connection 1-forms for this frame, such that

$$\nabla_X E_i = {\omega_i}^j(X)E_j$$

for all $X \in TM$.

B. Prove Cartan's first structure equation:

$$d\phi^j = \phi^i \wedge {\omega_i}^j + \tau^i$$

where $\{\tau^i\}_i$ are the torsion 2-forms defined in terms of the torsion tensor T by $T(X,Y) = \tau^j(X,Y)E_j$.

C. Suppose ∇ is the Levi-Civita connection on (M, g). Define the matrix of curvature 2-forms by

$${\Omega_i}^j = \frac{1}{2}{R_{kli}}^j \phi^k \wedge \phi^l,$$

and show they satisfy Cartan's second structural equation:

$${\Omega_i}^j = d{\omega_i}^j - {\omega_i}^k \wedge {\omega_k}^j.$$