Math 60670 Homework 5

Due Friday, March 12.

Problem 1: A. Show that isometries between Riemannian manifolds take geodesics to geodesics.

B. Consider the half-space model of hyperbolic space $\mathbb{H}^2$, i.e. $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ equipped with the hyperbolic metric $g = (dx^2 + dy^2)/y^2$. Show that the geodesics of $(\mathbb{R}^2_+, g)$ are vertical half-lines and half-circles that intersect the "boundary" $\{y = 0\}$ orthogonally. Hint: Use part A and HW2 Problem 3.

C. Deduce $\mathbb{H}^2$ is complete.

Problem 2: A curve $\gamma : [0, b) \to M$ is said to converge to infinity if for every compact set K, there is a T so that $\gamma(t) \notin K$ for $t > T$ (i.e. γ converges to infinity in the one-point compactification of M at infinity). Show that a Riemannian manifold (M, g) is complete if and only if every regular curve that converges to infinity has infinite length.

Problem 3: A. Compute the components of the Riemann curvature tensor in coordinates (x^i) in terms of the Christoffel symbols.

B. Show that if (x^i) are normal coordinates centered at p, then

$$R_{ijkl}|_p = \frac{1}{2}(\partial_i \partial_l g_{ik} + \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}).$$

Problem 4: Prove the second Bianchi identity

$$(\nabla_T R)(X, Y, Z, W) + (\nabla_X R)(Y, T, Z, W) + (\nabla_Y R)(T, X, Z, W) = 0.$$

Hint: use normal coordinates.

Problem 5: Show the scalar curvature Scal at a point $p \in M$ can be written

$$\mathrm{Scal}(p) = \frac{n}{|S^{n-1}|} \int_{S^{n-1}} \mathrm{Ric}_p(\theta, \theta) d\theta,$$

where S^{n-1} is the Euclidean $(n-1)$-sphere, and $|S^{n-1}|$ is its $(n-1)$-volume.