# Math 60670 Homework 2

Due Friday, February 19th.

**Problem 1:** Show that any smooth, connected, 1-dimensional Riemannian manifold is isometric to one of $(0, L)$, $(0, \infty)$, $\mathbb{R}$, $\mathbb{R}/L\mathbb{Z}$, for some $L$, where each of these manifolds inherits the metric $ds^2 = dx^2$ from $\mathbb{R}$. You may take for granted that any smooth, connected, 1-dimensional manifold is diffeomorphic to $(0, 1)$ or $S^1$ (though proving this is a nice exercise).

**Problem 2:** (Lee 3.9) Let $(M, g)$ be an oriented Riemannian $n$-manifold, and let $(x^1, \ldots, x^n)$ be coordinates compatible with the orientation in the sense that $(\partial_{x^1}, \ldots, \partial_{x^n})$ is a positively-oriented basis. Show that the volume form $dV = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$.

**Problem 3:** (Lee problem 3-7(a)) Let $\mathbf{U}^2$ denote the hyperbolic plane, i.e. the upper half-plane in $\mathbb{R}^2$ with metric $h = (dx^2 + dy^2)/y^2$. Let $\mathrm{SL}(2, \mathbb{R})$ denote the group of $2 \times 2$ real matrices of determinant 1, and define an action of $A \in \mathrm{SL}(2, \mathbb{R})$ on points $z = x + iy \in \mathbf{U}^2 \subset \mathbb{C}$ by

$$A \cdot z = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Show this defines a smooth action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbf{U}^2$ by isometries of the hyperbolic metric.

**Problem 4:** (Lee problem 3.10) Two metrics $g_1, g_2$ on $M$ are called conformal if $g_1 = f g_2$ for some positive $f \in C^\infty(M)$. A diffeomorphism $\phi : (M_1, g_1) \to (M_2, g_2)$ is called conformal if the pullback $\phi^* g_2$ is conformal to $g_1$. Show that two metrics are conformal iff they define the same angles but not necessarily the same lengths. Show a diffeomorphim is conformal iff it preserves angles.