# Math 60670 Homework 1

Due Friday, February 12.

**Problem 1:** Let $V$ be an $n$-dimensional vector spaces. Prove that the the space $(1,1)$-tensors on $V$ is naturally (i.e. independent of basis) isomorphic to the space of endomorphisms of $V$.

**Problem 2:** Show that $T$ is a smooth $(k,l)$-tensor field on $M$ if and only if $T$ is a smooth, $\mathbb{R}$-multilinear function from $k$ vector fields and $l$ 1-forms to $\mathbb{R}$, which is linear over $C^\infty(M)$. By "smooth" we mean that if $X_1, \ldots, X_k \in \mathcal{X}(M)$, $\omega_1, \ldots, \omega_l \in \mathcal{X}^*(M)$, then $T(X_1, \ldots, X_k, \omega_1, \ldots, \omega_l) \in C^\infty(M)$.

**Problem 3:** Let $(x^i)$, $(y^\alpha)$ be local coordinates defined in some $U \subset M$. Suppose $A \in \mathcal{T}^{(2,1)}(M)$ in the $x$-coordinate system can be expressed as

$$A = A^k_{ij}(x) dx^i \otimes dx^j \otimes \partial_{x^k}.$$

Show that in the $y$-coordinate system the components of $A$ are

$$A^c_{ab}(y = y(x)) = \frac{\partial y^c}{\partial x^k} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} A^k_{ij}(x).$$

Use this to show explicitly that the result of contracting the $k$, $i$ indices together is independent of choice of coordinates.

**Problem 4:** Prove that the Lie bracket and Lie derivative coincide, i.e. if $X, Y \in \mathcal{X}(M)$, and $\phi_t(p)$ is the flow of $X$, then show

$$[X, Y](p) = \lim_{t \to 0} \frac{D\phi_{-t} Y(\phi_t(p)) - Y(p)}{t}.$$

Hint: work in coordinates in which $\phi_t(x^1, \ldots, x^n) = (x^1 + t, x^2, \ldots, x^n)$.

**Problem 5:** Construct an isometric embedding of a flat 2-torus into $\mathbb{R}^4$. Not for credit: Do you think this torus can be isometrically embedded into $\mathbb{R}^3$? You may find the "Nash embedding theorem" interesting.