Notes on the energy-drop covering argument of Naber-Valtorta

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1 recall

NOTE: for the duration of this note we will define

$$D^k_\mu(x, \rho) = \rho^{-k-2} \inf_{L^k} \int_{B_\rho(x)} d^2(z, L) d\mu(z).$$

In particular, we have *no* lower-boundedness assumption on μ.

We live in R^N. As before we set $r_\alpha = 2^{-\alpha}$, using Greek indices to denote scale 1/2. Recall the discrete Reifenberg theorem:

Theorem 1.1 (discrete reifenberg)**.** *Let $\{B_{r_p}(x_p)\}_p$ be a collection of disjoint balls with $r_p \leq 1$. Set*

$$\mu = \sum_p r_p^k \delta_{x_p}.$$

Suppose for any $x \in B_2$, and any $\alpha_0 \in \{0, 1, 2, \ldots\}$, we knoew

$$\sum_{\alpha \geq \alpha_0} \int_{B_{2r_{\alpha_0}}(x)} D^k_\mu(z, 16r_\alpha) d\mu(z) \leq r^k_{\alpha_0} \delta^2.$$

Then, provided $\delta \leq \delta_0(n)$, we have

$$\mu B_1 = \sum_{x_p \in B_1} r^k_p \leq C_{dr}(n).$$

We will be considering a stationary integral n-varifold I in Euclidean space R^N. We denote the mass density by $\theta_r(x) = r^{-n}||I||(B_r(x))$. Recall $\theta_r(x)$ is monotone increasing in r; the exact formula is given in section "relating distoration and density."

We define the rescaled varifold $I_{y,\rho}$ by

$$I_{y,\rho}(A) = \rho^{-n} I(\{(y + \rho x, S) : (y, S) \in A\}).$$

Notice this agrees with the notation of "naber2.pdf" in the sense that

$$\mu_{(I_{y,\rho})} = (\mu_I)_{y,\rho}.$$

The stratum S^k is defined by

$$\begin{aligned} S^k &= \{x : \text{every tangent cone at } x \text{ has } \dim \text{spine} \leq k\} \\ &= \{x : \text{no tangent cone at } x \text{ has } \dim \text{spine} \geq k+1\}. \end{aligned}$$

By spine we simply mean "maximal subspace of translational symmetry."

We use two quantifications of S^k. The k, ϵ-stratum striates S^k by "how far" each tangent cone is from havine $k+1$-degrees of translational symmetry.

$$S^k_\epsilon = \{x : I \llcorner B_r(x) \text{ is not } (k+1, \epsilon)\text{-symmetric, for any } r \in (0, 1)\}.$$

Here we say that "$I \llcorner B_r(x)$ is (ℓ, ϵ)-symmetric" if there is some cone $\tilde{I}$, with $\dim \text{spine} \tilde{I} \geq \ell$, so that

$$d_{B_r(x)}(I, \tilde{I}) < \epsilon,$$

where $d_{B_r(x)}$ is an appropriately scaled choice of metric induced by varifold convergence.

To be precise, we could define $d_{B_r(x)}$ as follows. Let ϕ_i be a dense set in $C^0_c(B_2 \times \text{Gr}(n, N))$. Set

$$d_{B_r(x)}(I_1, I_2) = \sup_{A \in O(N)} \sum_i 2^{-i} \min\{1, |((I_1)_{x,r} \llcorner B_1)(\phi_i \circ A) - (I_2)_{x,r} \llcorner B_1)(\phi_i \circ A)|\}.$$

where we write

$$(I \llcorner B_1)(\phi) := \int_{B_1 \times Gr(n,N)} \phi(x, S) dI(x, S).$$

The supremum is to ensure the metric is invariant under rotations.

We have $I_i \llcorner B_r(x) \to I \llcorner B_r(x)$ as varifolds iff $d_{B_r(x)}(I_i, I) \to 0$. Also notice $d_{B_r(x)}$ is trivially scale-invariant, in the sense that

$$d_{B_r(x)}(I_1, I_2) = d_{B_1}((I_1)_{x,r}, (I_2)_{x,r}).$$

The k, ϵ, r-stratum further seperates each S^k_ϵ based on the "scale" at which points are ϵ-far from being $k+1$-symmetric. Specifically:

$$S^k_{\epsilon,r} = \{x : I \llcorner B_s(x) \text{ is not } (k+1, \epsilon)\text{-symmetric for any } s \in (r, 1)\}.$$

Trivially we have $S^k_{\epsilon,s} \subset S^k_{\epsilon,r}$ for $s \leq r$, and $S^k_\delta \supset S^k_\epsilon$ for $\delta \leq \epsilon$. Clear also is the identity $S^k_\epsilon = \cap_r S^k_{\epsilon,r}$. ("the k, ϵ-strata are ϵ-far at all scales")

We show that $S^k = \cup_\epsilon S^k_\epsilon$. ("each point in the k-stratum is a fixed distance from being $k+1$-symmetric") Suppose the contrary: there is a point $x \in S^k$, and a sequence of tangent cones C_i so that C_i is $(k+1, 1/i)$-symmetric. Take a subsequential limit we and obtain a tangent cone C at x, which is $k+1$-symmetric. This is a contradiction.

2 the lemma

We work towards proving the following covering lemma. It is the key ingredient to proving Minkowski bounds and rectifiability of the singular set.

Theorem 2.1 (lemma 5.1 in N-V)**.** *Let I be an integral stationary m-varifold in B_8, with $||I||(B_8) \leq \Lambda$. Write $E = \sup_{B_1} \theta_1$, and take $\epsilon > 0$.*

For any $\eta \leq \eta_2(\Lambda, \epsilon, n)$, and any $r > 0$, we can find collections

$$U_r = \{B_r(x_i)\}, \quad U_+ = \{B_{r_i}(y_i)\} \quad (r_i \geq r),$$

so that $\{B_{r/5}(x_i)\}$ are disjoint, and $\{B_{r_i/5}(y_i)\}$ are disjoint, and every $x_i, y_i \in S^k_{\epsilon,r} \cap B_1$.

The collections U_r and U_+ enjoy the following properties:

1. all balls $U_r \cup U_+$ form a cover of $S^k_{\epsilon,r} \cap B_1$.

2. for each ball in U_+ we have fixed density drop: $\sup_{B_{r_i}(y_i)} \theta_{r_i} \leq E - \eta$.

3. we have mass bound $\#\{x_i\} r^k + \sum_i r_i^k \leq C_{cover}(\Lambda, \epsilon, n)$

The key link is theorem 4.1, which says we can bound the distortion in terms of density drop. Therefore, small density drop at the ball centers will give us good distoration bounds, so as to apply discrete Reifenberg. A major pain in the ass we will encounter is that, although it's easy obtain nice density drop "nearby", obtaining good density drop at the centers themselves is very delicate.

To prove the Minkowski bounds of $S^k_{\epsilon,r}$, i.e.

$$|B_r(S^k_{\epsilon,r})| \leq C_\epsilon r^{n-k},$$

we use lemma 5.1 inductively. First observe this follows immediately if $U_+ = \emptyset$. In general, we apply lemma 5.1 initially to $B_1 \cap S^k_{\epsilon,r}$, then again in each ball of

U_+, then again, etc. Each time we will have a fixed density drop, and hence the process must terminate in finitely many steps. In the end we won't have any U_+ balls.

The setup for proving lemma 5.1 is pretty self-contained. All the hard work is in establishing the mass bound, point 3. We start the proof.

Start of proof. Define

$$s_x = s_x(E, \eta) = \sqrt{10/11} \inf\{r \leq s \leq 1 : \sup_{B_s(x) \cap S^k_{\epsilon,r}} \theta_s \geq E - \eta\}.$$

This definition may seem a little opaque at first. The point in some sense is to establish the biggest ball with a fixed drop. One consequence of this definition is that we can always find some $y_x \in S^k_{\epsilon,r} \cap B_{11s_x/10}(s)$ with small density drop, in the sense that $\theta_{11s_x/10}(y_x) \geq E - \eta$. The extra factor in front of s_x is just the density drop may not be achieved precisely at s_x; we have to shrink our ball a little. As mentioned above, a huge nuisance is that y_x need not coincide with x.

If $s_x > \sqrt{10/11}r$, then we obtain the infimum, and hence $\sup_{B_{s_x}(x)} \theta_{s_x} \leq E - \eta$. In other words, on $B_{s_x}(x)$ we have a fixed drop in density. The quantity s_x is called the 'energy scale', and is how we will stratify our scales.

We have the cover $\{B_{s_x}(x)\}_{x \in S^k_{\epsilon,r} \cap B_1}$. We let U_r be a sub-cover $\{B_r(x_i)\}_i$ of

$$\{B_r(x) : x \in S^k_{\epsilon,r} \cap B_1, s_x = r\},$$

with the property that that $\{B_{r/5}(x_i)\}_i$ are disjoint. Similarly, take $U_+ = \{B_{r_i}(y_i)\}_i$ to be a sub-cover of

$$\{B_{s_x}(x) : x \in S^k_{\epsilon,r} \cap B_1, s_x > r\},$$

with the property that $\{B_{r_i/5}(y_i)\}_i$ are disjoint. (to obtain U_r and U_+, actually look at the cover $\{B_{s_x/5}(x)\}$, and then choose a Vitali subcovers)

We have by construction that on each ball $B_{r_i}(y_i)$ in U_+, we have $\sup_{B_{r_i}(y_i)} \theta_{r_i} \leq E - \eta$. Of course we also know that $U_r \cup U_+$ form a cover of $S^k_{\epsilon,r} \cap B_1$.

Now comes the hard part. For each x define the integer $\alpha(x)$ by $r_{\alpha(x)} \leq s_x < 2r_{\alpha(x)}$. Now set

$$U'_r = \{B_{r_{\alpha(x_i)}}(x_i)\}, \quad U'_+ = \{B_{r_{\alpha(y_i)}}(y_i)\}.$$

Each U'_r and U'_+ is a collection of balls $\{B_{r_{\alpha_i}}(z_i)\}$ with the following properties:

$$\{B_{r_{\alpha_i}/5}(z_i)\} \text{ are disjoint}, \qquad \sup_{B_{11r_{\alpha_i}/5}(z_i)} \theta_{11r_{\alpha_i}/5} \geq E - \eta, \quad z_i \in S^k_{\epsilon,r}, \quad r_{\alpha_i} \geq r/8.$$

(of course since $S^k_{\epsilon,s} \subset S^k_{\epsilon r}$ for $s \leq r$, we can WLOG assume r is a power of 2). From these properties we wish to deduce mass bounds. This will follow by the very last theorem general-mass-bounds. The entirety of the rest of this note is devoted towards proving this last theorem. □

3 quantitative stratification

The first Lemma is a basic contradiction argument, using monotonicity.

Lemma 3.1. *For any $\eta' > 0$, there is an $R_1(\Lambda, \eta', n)$ so that, if $R \geq R_1$ and $\eta \leq \eta'/10$, then*

$$\sup_{B_r(x)} \theta_r \geq E - \eta \implies \theta_{Rr}(x) \geq E - \eta'.$$

We also want to know how to recenter density.

Lemma 3.2. *Suppose there is a $y \in B_1$ so that $\theta_1(y) \geq E - \eta$. Then provided $R \geq R_0(E, \eta, n)$, we have*

$$\theta_R(0) \geq E - 2\eta.$$

Proof. We have

$$\begin{aligned}
\Theta(0, R) &= \frac{\mu_I(B_R(0))}{R^n} \\
&\geq \frac{\mu_I(B_{R-1}(y))}{(R-1)^n}(1 - 1/R)^n \\
&\geq (E - \eta)(1 - 1/R)^n \\
&\geq E - 2\eta
\end{aligned}$$

for R sufficiently big. □

The second is a slightly more complicated contradiction argument, an effective version of dimension reduction.

Lemma 3.3. *Suppose* $\sup_{B_1} \theta_1(z) \leq E$*, and there are points $x_0, \ldots, x_k$ in B_1 satisfying:*

$$x_i \notin B_\tau(< x_0, \ldots, x_{i-1} >) \quad \forall i, \quad \theta_\eta(x_i) \geq E - \eta.$$

Then provided $\eta \leq \eta_0(E, \tau, \eta', \epsilon)$, we must have either

$$\theta_{1/40}(0) \geq E - \eta',$$

or $I \llcorner B_\beta(0)$ is $(k+1, \epsilon)$-symmetric, for some $\beta = \beta(E, \tau, \eta', \epsilon)$.

Proof. We prove this by a series of contradiction arguments.

Claim 1: provided η_0 is sufficiently small, depending only on (τ, η', E), then for some $\alpha = \alpha(\tau, \eta', E)$ we have that

$$\theta_{1/40}(0) < E - \eta' \quad \implies \quad 0 \notin B_\alpha(< x_0, \ldots, x_k >).$$

Otherwise, take a counter example sequence I_i, satisfying the hypothesis with $\eta = \alpha = 1/i$, but $\theta_{1/40}(I_i, 0) < E - \eta'$. Passing to a subsequence, we get an I, and points $x_0, \ldots, x_k$ in τ-general position which:

1. satisfy $\theta_0(x_i) = E$,
2. $0 \in < x_0, \ldots, x_k >$,
3. $\theta_{1/40}(0) \leq E - \eta$.

But since I is necessarily a cone over each x_i, we have $\theta_0(0) = E$ also, contradicting monotonicity. This proves the first claim.

Claim 2: taking $\alpha(\tau, \eta', E)$ as above, and provided η_0 is small (depending on τ, η', E, ϵ), we have that

$$0 \notin B_\alpha(< x_0, \ldots, x_k >) \quad \implies \quad B_\beta(0) \text{ is } (k+1, \epsilon)\text{-symmetric},$$

for some $\beta = \beta(\tau, \eta', E, \epsilon)$.

Otherwise, choose a counter example sequence I_i with $\eta = \beta = 1/i$. Pass to a subsequence, we get a limit I which $(k, 0)$-symmetric WRT some V^k, so that $0 \notin B_\alpha(V)$. But then I claim that for all r sufficiently small, $I \llcorner B_r(0)$ is $(k+1, \epsilon/2)$-symmetric. This follows because any blow-up at 0 must be a $(k+1, 0)$-symmetric cone.

But then $I_i \llcorner B_{1/i}(0)$ is $(k+1, \epsilon)$-symmetric for large i. This is a contradiction.

□

4 relating distoration and density

We work towards proving the following theorem (theorem 4.1 in N-V):

Theorem 4.1. *Let I be a stationary n-varifold in B_{8r}, with $r^{-n}||I||(B_{8r}) \leq \Lambda$. For any $\epsilon > 0$, there are $\delta(\epsilon, \Lambda)$ and $c(\epsilon, \Lambda)$ so that if*

$$\begin{cases} I \llcorner B_{8r} \text{ is } (0, \delta)\text{-symmetric} \\ I \llcorner B_{8r} \text{ is not } (k+1, \epsilon)\text{-symmetric} \end{cases},$$

then for any finite measure μ we have

$$r^{-k-2} \inf_{L^k} \int_{B_r} d(z, L)^2 d\mu(z). \leq \frac{C}{r^k} \int_{B_r} \theta_{8r}(x) - \theta_r(x) d\mu(x).$$

Remark 4.2. The LHS coincides with our "new" definition $D^k_\mu(0, r)$. This is basically why we've redefined D^k_μ in this note.

Remark 4.3 (stolen from Otis)**.** Here's is some intuition about why we need no assumptions on μ. Suppose the RHS $\equiv 0$, and for simplicity take $r = 1$. Then at each point $x \in \text{spt}\mu$, I must be 0-symmetric in $A_{1,8}(x)$. In particular, if there are $k+1$ linearly indepedent points in $\text{spt}\mu \cap B_1$, then I must be $(k+1)$-symmetric in $A_{3,4}$. But then by $(0, \delta)$-symmetry, we have B_8 is $(k+1, \epsilon_2(\delta))$-symmetric, with $\epsilon_2 \to 0$ as $\delta \to 0$. Choosing δ sufficiently small gives a contradiction, and therefore $\text{spt}\mu \cap B_1 \subset k$-plane.

4.4 two claims

We fix μ a finite measure supported in B_r, and let m be the μ-center of mass. Define the symmetric bilinear form

$$Q(v, w) = \fint < z - m, v >< z - m, w > d\mu(z).$$

Let v_k be an ON eigenbasis, with associated eigenvalues λ_k ordered so that $\lambda_1 \geq \ldots \geq \lambda_n$.

Proposition 4.5. *Let I be a stationary integral n-varifold in B_{8r}. We have*

$$\lambda_k r^{-n-2} \int_{A_{3r,4r}} | < I_z^\perp, v_k > |^2 d||I||(z) \leq c(n) \fint \theta_{8r}(x) - \theta_r(x) d\mu(x).$$

Here $| < I_z^\perp, v_k > |^2 = \sum_i | < e_i, v_k > |^2$, where e_i is an ON basis of the normal space $I_z^\perp$ (defined $||I||$-a.e.).

Proof. For a given z, choose an ON frame $\{e_i(z)\}$ for $I_z^\perp$, and deduce

$$\begin{aligned}
\lambda_k < e_i(z), v_k > &= Q(e_i(z), v_k) \\
&= \fint < e_i(z), x - m >< v_k, x - m > d\mu(x) \\
&= \fint < e_i(z), x - m - (z - m) >< v_k, x - m > d\mu(x).
\end{aligned}$$

Therefore by Holder we have for any z

$$\lambda_k | < I_z^\perp, v_k > |^2 \leq \fint | < I_z^\perp, z - x > |^2 d\mu(x).$$

Recall that

$$\begin{aligned}
\theta_{8r}(x) - \theta_r(x) &= 2 \int_{A_{r,8r}(x)} r_x^{-n} |D^\perp r_x|^2 d||I||(z) \\
&= 2 \int_{A_{r,8r}(x)} |z - x|^{-n} \left| < I_z^\perp, \frac{z - x}{|z - x|} > \right|^2 d||I||(z).
\end{aligned}$$

Since $\mathrm{spt}\mu \subset B_r$, we have

$$\begin{aligned}
&\lambda_k r^{-n-2} \int_{A_{3r,4r}(0)} | < I_z^\perp, v_k > |^2 d||I||(z) \\
&\leq r^{-n-2} \int_{A_{3r,4r}} \fint | < I_z^\perp, z - x > |^2 d\mu(x) d||I||(z) \\
&\leq 5^n \fint \int_{A_{3r,4r}} | < I_z^\perp, z - x > |^2 |z - x|^{-n-2} d||I||(z) d\mu(x) \\
&\leq 5^n \fint \int_{A_{r,8r}(x)} | < I_z^\perp, z - x > |^2 |z - x|^{-n-2} d||I||(z) d\mu(x) \\
&= c(n) \fint \theta_{8r}(x) - \theta_r(x) d\mu(x).
\end{aligned}$$

□

Proposition 4.6. *Let I be a stationary integral n-varifold in B_{8r}, with $r^{-n}||I||(B_{8r}) \leq \Lambda$. For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon, \Lambda)$ so that if*

$$\begin{cases} I \llcorner B_{8r} \text{ is } (0,\delta)\text{-symmetric} \\ I \llcorner B_{8r} \text{ is not } (k+1,\epsilon)\text{-symmetric} \end{cases},$$

then

$$r^{-n} \int_{A_{3r,4r}} |< I_z^{\perp}, V >|^2 d||I||(z) \geq \delta$$

for any $(k+1)$-dimensional space V. Here we write $|< I_z^{\perp}, V >|^2 = \sum_i |< I_z^{\perp}, e_i >|^2$, where $\{e_i\}$ is any ON basis of V.

Proof. Notice all quantities are scale-invariant, so we can always take $r \equiv 1$. Suppose the theorem is false. Then we have a sequence $I(i)$ of stationary varifolds, and a sequence δ_i, so that

$$\begin{cases} I(i) \llcorner B_8 \text{ is } (0,\delta_i)\text{-symmetric} \\ I(i) \llcorner B_8 \text{ is not } (k+1,\epsilon)\text{-symmetric} \end{cases},$$

but for some $k+1$ plane $V(i)$ we have

$$\int_{A_{3,4}} |< I(i)_z^{\perp}, V(i) >|^2 d||I(i)||(z) < \delta_i.$$

Since we have uniform mass bounds on B_8, we can take WLOG suppose $I(i) \to I$ as varifolds, where I is stationary and has $||I||(B_8) \leq \Lambda$. Similarly, we can suppose $V(i) \to V$ for some $k+1$ plane V. So varifold convergence gives us

$$\int_{A_{3,4}} |< I_z^{\perp}, V >|^2 d||I||(z) = 0.$$

But since $I \llcorner B_8$ is 0-symmetric, this implies in fact that

$$\int_{B_8} |< I_z^{\perp}, V >|^2 d||I||(z) = 0.$$

In particular, for large enough i we will have $I(i)$ being $(k+1,\epsilon)$-symmetric, which is a contradiction. □

4.7 proving the theorem

Proof of theorem. Of course we can assume $\mu = \mu \llcorner B_r$. Let m, $\{v_k\}$ and $\{\lambda_k\}$ be as before. Recall that $\lambda_1 \geq \ldots \geq \lambda_n$. For ease of notation let's just take the LHS to be $D_\mu^k(0,r)$, it really doesn't matter.

One can easily verify that the L^2-best plane $V_\mu^k(0,1) = m + \text{span}\{v_1, \ldots, v_k\}$, and in particular

$$D_\mu^k(0,r) = \frac{\mu B_r}{r^{k+2}} (\lambda_{k+1} + \ldots + \lambda_n).$$

Choose δ as in proposition energy-concentration. By hypothesis, we can apply propositions energy-drop-evectors and energy-concentration do deduce that inequalities

$$\begin{aligned}
&c(n) \fint_{B_1} \theta_{8r}(x) - \theta_r(x) d\mu(x) \\
&\geq \frac{1}{k+1} r^{-n-2} \sum_{i=1}^{k+1} \lambda_k \int_{A_{3,4}} | < I_z^\perp, v_i > |^2 d||I||(z) \\
&\geq \frac{1}{k+1} r^{-2} \lambda_{k+1} r^{-n} \int_{A_{3,4}} | < I_z^\perp, \mathrm{span}\{v_1, \ldots, v_{k+1}\} > |^2 d||I||(z) \\
&\geq \frac{1}{(n-k)(k+1)} r^{-2} (\lambda_{k+1} + \ldots + \lambda_n) \delta \\
&= \frac{\delta}{c(n)} \frac{r^k}{\mu B_r} D^k_\mu(0, r).
\end{aligned}$$

□

5 mass bounds from good density drop

We wish to use small density drop with theorem 4.1 and discrete-reifenberg to obtain mass bounds. We first give a baby case to illustrate the idea.

5.1 a baby case

Lemma 5.2 (baby density-to-mass bounds)**.** *Let $\{B_r(x_i)\}$ be a collection of disjoint balls, satisfying:*

$$\theta_{\gamma r}(x_i) \geq E - \eta, \quad x_i \in S^k_{\epsilon,r} \cap B_1,$$

where $E = \sup_{x \in B_1} \theta_1(x)$. We assume $||I||(B_8) \leq \Lambda$, and take $\gamma(\Lambda, \epsilon, n) < 1$ as in theorem quant-cone (theorem 2.1 in N-V).

Then provided η is sufficiently small, depending only on (Λ, ϵ, n), we have

$$\#\{x_i\} \leq C(n) r^{-k}.$$

Proof. Define $\mu = r^k \sum_i \delta_{x_i}$. Let α_1 be the smallest integer so $r_{\alpha_1} \leq r$. We prove inductively that

$$\mu B_{r_\alpha}(x) \leq C_{dr}(n) r_\alpha^k$$

for every $\alpha \in [\alpha_1, 4]$. Here $C_{dr}(n)$ is the constant from theorem discrete-reifenberg. Of course this holds trivally for $\alpha = \alpha_1$, since $C_{dr} \geq 2^k$. Having proven this inductively the theorem will follow by a trivial packing argument.

Suppose we have volume bounds up to $r_{\beta+1}$. So, for any α so that $r_{\alpha_1} \leq r_\alpha \leq r_{\beta+1}$, and any $x \in B_1$, we have

$$\mu B_{r_\alpha}(x) \leq C_{dr} r_\alpha^k.$$

OK given the above, we can obtain shitty volume bounds a few scales up by simple packing arguments. In particular, we have a bound like

$$\mu B_s(x) \leq C_2(n) s^k.$$

for any $s \in [r_{\alpha_1}, 18 r_\beta]$, where $C_2 >> C_{dr}$.

We now obtain the right distortion bounds to apply theorem discrete-reifenberg at scale r_β. Take $x \in B_1$. We estimate, using Fubini's theorem and theorem 4.1

$$\begin{aligned}
&\sum_{r_\alpha \leq 16 r_\beta} \int_{B_{2r_\beta}(x)} D(z, r_\alpha) d\mu(z) \\
&\leq c \sum_{\alpha_1 > \alpha \geq \beta - 4} \frac{1}{r_\alpha^k} \int_{B_{2r_\beta(x)}} \int_{B_{r_\alpha}(z)} W_{r_\alpha}(y) d\mu(y) d\mu(z) \\
&= c \sum_\alpha r_\alpha^{-k} \int_{B_{2r_\beta + r_\alpha}(x)} \mu(B_{r_\alpha}(y) \cap B_{2r_\beta}(x)) W_{r_\alpha}(y) d\mu(y) \\
&\leq c C_2 \int_{B_{18 r_\beta}(x)} \sum_{\alpha_1 > \alpha \geq \beta - 4} W_{r_\alpha}(y) d\mu(y) \\
&\leq 3 c C_2 \int_{B_{18 r_\beta}(x)} \theta_{16 r_\beta}(y) - \theta_r(y) d\mu(y) \\
&\leq 3 c(\Lambda, \epsilon) (C_2)^2 (r_\beta)^k \eta.
\end{aligned}$$

Choose η sufficiently small, depending only on Λ, ϵ, n, so we have

$$\sum_{r_\alpha \leq 16 r_\beta} \int_{B_{2r_\beta}(x)} D(z, r_\alpha) d\mu(z) \leq (r_\beta)^k \tau^2$$

where τ is sufficiently small to apply theorem discrete-reifenberg. Having done this, we obtain

$$\mu B_{r_\beta}(x) \leq C_{dr} r_\beta^k.$$

□

5.3 the general case

We now prove the theorem in full glory.

Theorem 5.4 (density to mass bounds)**.** *Let $\{B_{2r_{\alpha_i}}(x_i)\}$ be a collection of disjoint balls, so that:*

$$\theta_{128 r_{\alpha_i}}(x_i) \geq E - \eta, \quad r \leq r_{\alpha_i} \leq 1, \quad x_i \in S^k_{\epsilon, r} \cap B_1$$

Here $E = \sup_{x \in B_1} \theta_1(x)$, and we assume $\theta_8(0) \leq \Lambda$.

Provided $\eta \leq \eta_0(\Lambda, \epsilon, n)$ we have

$$\sum_i r_{\alpha_i}^k \leq C_{mb}(\Lambda, \epsilon, n)$$

Proof. Let

$$\mu_\alpha = \sum_{\alpha_i \geq \alpha} r_{\alpha_i}^k \delta_{x_i}$$

For ease of notation write $D_\alpha = D^k_{\mu_\alpha}$.

We make the following crucial remark: if $\alpha > \beta$, then by our disjointness hypothesis we have *either* $\mathrm{spt}\mu_\beta \cap B_{r_\alpha}(x)$ consists of a single point, *or* $\mu_\beta B_{r_\alpha}(x) = \mu_\alpha B_{r_\alpha}(x)$. In the former case we have $D_\beta(x, r_\alpha) = 0$, and in the latter we have $D_\beta(x, r_\alpha) = D_\alpha(x, r_\alpha)$.

Similarly, if $\alpha \geq \alpha_i$, then we have $D_\beta(x_i, r_\alpha) = 0$ for any β. This follows because either $\mathrm{spt}\mu_\beta \cap B_{r_\alpha}(x_i)$ is empty or a single point.

Conversely, if $\alpha < \alpha_i$, then for any β we have by lemma 4.1

$$\begin{aligned} D_\beta(x_i, r_\alpha) &\leq \gamma^{-k-2} D_\beta(x_i, \gamma^{-1} r_\alpha) \\ &\leq C(\Lambda, \epsilon, n) r_\alpha^{-k} \int_{B_{\gamma^{-1} r_\alpha}(x_i)} W_{\gamma^{-1} r_\alpha}(z) d\mu_\beta(z). \end{aligned}$$

Here we've chosen $\gamma(\Lambda, \epsilon, n)$ small, and ensured η_0 is small, to apply theorem 2.1 to deduce that $B_{8\gamma^{-1} r_\alpha}(x_i)$ is $(0, \delta)$-symmetric. Trivially $B_{8\gamma^{-1} r_\alpha}(x_i)$ is not $(k+1, \epsilon)$-symmetric, and $\theta_{8\gamma^{-1} r_\alpha}(x_i) \leq \Lambda$ follows by monotonicity. So we are justified in appling lemma 4.1.

Let $\alpha_1 = \max_i \alpha_i$. We work towards the estimate

$$(\dagger_\beta) \quad \mu_\beta B_{r_\beta}(x) \leq C_{dr}(n) r_\beta^k$$

for any $x \in B_1$, and any $r_\beta \leq \gamma 2^{-7}$. Here C_{dr} is the constant from the discrete Reifenberg theorem. Of course $(\dagger_{\alpha_1})$ holds trivially.

Given this, by our disjointness hypothesis the theorem will follow by a direct packing argument, with $C_{mb} = C(n, \gamma) C_{dr}$.

Suppose $(\dagger_{\beta+1})$ holds. By a packing argument, we can obtain a shitty bound like

$$\mu_\alpha B_s(x) \leq C_2(n, \gamma) s^k$$

for any $\alpha \geq \beta + 1$, and any $s \in [r_\alpha, 2\gamma^{-1} r_\alpha]$, with $C_2 >> C_{dr}$.

We have

$$\begin{aligned} &\sum_{\alpha \geq \beta - 4} \int_{B_{2r_\beta}(x)} D_{\beta+1}(z, r_\alpha) d\mu_{\beta+1}(z) \\ &= \sum_{\alpha \geq \beta+1} \int_{B_{2r_\beta}(x)} D_\alpha(z, r_\alpha) d\mu_\alpha(z) + \sum_{\beta \geq \alpha \geq \beta - 4} \int_{B_{2r_\beta}(x)} D_{\beta+1}(z, r_\alpha) d\mu_{\beta+1}(z) \end{aligned}$$

Basically we need to split this up because we don't have a bound like $\mu_{\beta+1} B_{r_\alpha} \leq C r_\alpha^k$, for α significantly larger than β.

For ease on our eyes let's consider each of the two RHS terms separately. We calculate, using Fubini's theorem,

$$
\begin{aligned}
\text{(first RHS term)} &\leq c \sum_{\alpha \geq \beta+1} \int_{B_{2r_\beta}(x)} r_\alpha^{-k} \int_{B_{\gamma^{-1}r_\alpha}(z)} W_{\gamma^{-1}r_\alpha}(y) d\mu_\alpha(y) d\mu_\alpha(z) \\
&\leq c \sum_{\alpha \geq \beta+1} \int_{B_{2\gamma^{-1}r_\beta}(x)} C_2 W_{\gamma^{-1}r_\alpha}(y) d\mu_\alpha(y) \\
&= cC_2 \sum_{\alpha \geq \beta+1} \sum_{\substack{x_i \in B_{2\gamma^{-1}r_\beta}(x) \\ \alpha_i > \alpha}} W_{\gamma^{-1}r_\alpha}(x_i) r_{\alpha_i}^k \\
&\leq cC_2 \sum_{\substack{x_i \in B_{2\gamma^{-1}r_\beta}(x) \\ \alpha_i \geq \beta+1}} r_{\alpha_i}^k \sum_{\alpha_i > \alpha \geq \beta+1} W_{\gamma^{-1}r_\alpha}(x_i) \\
&\leq 3cC_2 \sum_{\substack{x_i \in B_{2\gamma^{-1}r_\beta}(x) \\ \alpha_i \geq \beta+1}} r_{\alpha_i}^k (\theta_{8\gamma^{-1}r_{\beta+1}}(x_i) - \theta_{2\gamma^{-1}r_{\alpha_i}}(x_i)) \\
&\leq 3c(C_2)^2 \eta r_\beta^k
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(second RHS term)} &\leq c \sum_{\beta \geq \alpha \geq \beta-4} \int_{B_{2r_\beta}(x)} r_\alpha^{-k} \int_{B_{\gamma^{-1}r_\alpha}(z)} W_{\gamma^{-1}r_\alpha}(y) d\mu_{\beta+1}(y) d\mu_{\beta+1}(z) \\
&\leq c \sum_{\beta \geq \alpha \geq \beta-4} \int_{B_{2\gamma^{-1}r_\beta}(x)} C_2 W_{\gamma^{-1}r_\alpha}(y) d\mu_{\beta+1}(y) \\
&= cC_2 \sum_{\substack{x_i \in B_{2\gamma^{-1}r_\beta}(x) \\ \alpha_i \geq \beta+1}} \sum_{\beta \geq \alpha \geq \beta-4} W_{\gamma^{-1}r_\alpha}(x_i) r_{\alpha_i}^k \\
&\leq 3cC_2 \sum_{\substack{x_i \in B_{2\gamma^{-1}r_\beta}(x) \\ \alpha_i \geq \beta+1}} r_{\alpha_i}^k (\theta_{8\gamma^{-1}r_{\beta-4}}(x_i) - \theta_{2\gamma^{-1}r_{\beta+1}}(x_i)) \\
&\leq 3c(C_2)^2 \eta r_\beta^k
\end{aligned}
$$

To be explicit, here's how we used Fubini's theorem:

$$
\begin{aligned}
\int_{B_1} \left(\int_{B_r(z)} f(y) d\mu(y) \right) d\nu(z) &= \int_{B_1} \left(\int_{B_{1+r}} 1_{B_r(z)}(y) f(y) d\mu(y) \right) d\nu(z) \\
&= \int_{B_{1+r}} f(y) \left(\int_{B_1} 1_{B_r(y) \cap B_1}(z) d\nu(z) \right) d\mu(y) \\
&\leq \int_{B_{1+r}} f(y) \nu(B_r(y)) d\mu(y).
\end{aligned}
$$

Now we can choose η_0 sufficiently small, depending only on ϵ, Λ, n, so we have the bound

$$\sum_{\alpha \geq \beta - 4} \int_{B_{2r_\beta}(x)} D_{\beta+1}(z, r_\alpha) d\mu_{\beta+1}(z) \leq r_\beta^k \tau^2$$

where τ is sufficiently small to apply theorem discrete-reifenberg. We therefore deduce that

$$\mu_{\beta+1} B_{r_\beta}(x) \leq C_{dr} r_\beta^k.$$

But by our crucial remark, we have either $\mu_\beta B_{r_\beta}(x) = r_\beta^k$, or $\mu_\beta B_{r_\beta}(x) = \mu_{\beta+1} B_{r_\beta}(x)$. Therefore we have proven $(\dagger_\beta)$. □

6 mass bounds of a general covering

Here is our strategy. We will show that if our covering has sufficiently large scale drop, then we have a kind of dichotomy: each ball either has small density drop at the center, or has small mass. Our covering is always such that we have small density drop somewhere, but the issue in general is recentering the balls while maintaining mass control. The aforementioned dichotomy allows us to do this.

In this section we write $r_i = \rho^i$. So, latin indices mean scale ρ.

Lemma 6.1. *Let μ be a Borel measure, and suppose $\mu(B_\tau(x)) \leq M\tau^k$ for all $x \in B_1$. Then we have: either are $x_0, \ldots, x_k \in \mathrm{spt}(\mu)$ with the property that*

$$x_i \notin B_\tau(< x_0, \ldots, x_{i-1} >) \quad \forall i,$$

or $\mu(B_1) \leq c_1(n) M\tau$.

Proof. If the conclusion fails, then we must have $\mathrm{spt}(\mu) \subset B_{2\tau}(V^{k-1})$ for some $(k-1)$ affine plane. Let $\{B_\tau(y_i)\}_i$ be a Vitali covering of $B_{2\tau}(V^{k-1})$. Then we have

$$\#\{y_i\}_i \leq c(n)\tau^{1-k}.$$ □

Lemma 6.2. *Take $\epsilon, \tau > 0$. Let $E = \sup_1 \theta_1(z)$. Let μ be a Borel measure with the following properties:*

A) $\mathrm{spt}\mu \subset S^k_{\epsilon, \beta r}$,

B) $\sup_{B_\rho(x)} \theta_\rho(x) \geq E - \eta$ *for every* $x \in \mathrm{spt}\mu$,

C) $\mu(B_\tau(x)) \leq M\tau^k$ *for every* $x \in B_1$.

Then provided $\eta \leq \eta_1(n, E, \eta', \tau, \epsilon)$, $\rho \leq \rho_1(n, E, \tau, \eta, \epsilon)$, $\beta \leq \beta_1(n, E, \eta', \tau, \epsilon)$, we have the following dichotomy: either

$$\theta_{1/40}(0) \geq E - \eta',$$

or

$$\mu(B_1) \leq c_1(n) M\tau.$$

Proof. Combine Lemmas 6.1 and 3.2, to deduce that either there are $x_0, \ldots, x_k \in B_1$ with the property that

$$x_i \notin B_\tau(< x_0, \ldots, x_i >) \quad \forall i,$$

and $\theta_{R\rho}(x_i) \geq E - 2\eta$, where $R = R(n, E, \eta)$. Or, if this fails, then $\mu(B_1) \leq c_1(n) M \tau$.

In the former case, ensure that $\rho \leq \eta_0/R$, and $\eta \leq \eta_0/2$, where η_0 as in Lemma 3.3. We can then apply Lemma 3.3 to make the desired conclusion. $\square$

Take $\tau = \frac{1}{c_1 M}$, $\eta' = \eta_0(n)$. Choose $\eta = \eta_1(n, E, \eta', \tau, \epsilon) \leq \eta'$, and $\rho \leq \rho_1(n, E, \eta, \tau, \epsilon)$, and $\beta = \beta_1(n, E, \eta, \epsilon)$.

Theorem 6.3. *Let $\{B_{r_{i_p}}(x_p)\}_p$ be a collection of disjoint balls, with the properties*

$$\sup_{B_{11 r_{i_p}}(x_p)} \theta_{11 r_{i_p}}(z) \geq E - \eta_1, \quad r_A \leq r_{i_p} \leq 1, \quad x_p \in B_1 \cap S^k_{\epsilon, \beta r}.$$

Then

$$\sum_p r^k_{i_p} \leq C_{gmb}(\Lambda, \epsilon, n).$$

Proof. Write

$$\mu_i = \sum_{r_{i_p} \leq r_i} r^k_{i_p} \delta_{x_p}$$

We prove inductively that

$$(\dagger_i) \quad \mu_i(B_r(x)) \leq M(\Lambda, \epsilon, n) r^k \quad \forall x \in B_1, \quad \forall r_i \leq r \leq 1.$$

Notice that $(\dagger_{A+1})$ is vacuously true. Let us now assume $(\dagger)$ holds up to scale $i+1$. We show $(\dagger_i)$ holds also.

Lemma 6.2 gives us the following dichotomy: either

$$\theta_{r_{i_p}}(x_p) \geq E - \eta_0,$$

or

$$\mu_i(B_{40 r_{i_p}}(x_p)) \leq c_2(n) r^k_{i_p}$$

This follows because by disjointness we know

$$\mu_i(B_{40 r_{i_p}}(x_p)) \leq \mu_{i+1}(B_{40 r_{i_p}}(x_p)) + c(n) r^k_{i_p},$$

and so we can apply Lemma 6.2 to the measure μ_{i+1} at scale $40 r_{i_p}$, recalling our choices of ρ, τ. This proves the dichotomy.

Fix some $B_r(x)$, with $x \in B_1$ and $r \geq r_i$. We will "recenter" the balls in $\mathrm{spt}\mu_i \cap B_r(x)$ to a collection $\{B_{\bar{r}_p}(\bar{x}_p)\}_p$ with good density drop, and good μ_i-measure control. This last point is crucial, and is the reason for the dichotomy snafu.

Take $x_p \in B_r(x) \cap \mathrm{spt}\mu_i$. If $\theta_{r_{i_p}}(x_p) \geq E - \eta_0$, then let $\bar{x}_p = x_p$, and $\bar{r}_p = r_{i_p}$. In this case

$$\mu_i(B_{\bar{r}_p}(\bar{x}_p)) = \mu_{i_p}(B_{r_{i_p}}(x_p)) = \bar{r}_p^k$$

by disjointness.

Otherwise, by assumption we can choose $\bar{x}_p \in B_{12r_{i_p}}(x_p)$ so that $\theta_{12r_{i_p}}(\bar{x}_p) \geq E - \eta_1 \geq E - \eta_0$. In this case take $\bar{r}_p = 12r_{i_p}$, and we deduce

$$\mu_i(B_{\bar{r}_p}(\bar{x}_p)) \leq \mu_i(B_{40r_{i_p}}(x_p)) \leq c_2(n)\bar{r}_p^k,$$

by our dichotomy.

Either way, we deduce the $\bar{x}_p$ satisfy:

$$\theta_{\bar{r}_p}(\bar{x}_p) \geq E - \eta_0, \quad \mu_i(B_{\bar{r}_p}(\bar{x}_p)) \leq c_2(n)\bar{r}_p^k, \quad \mathrm{spt}\mu_i \subset_p \cup B_{\bar{r}_p}(\bar{x}_p).$$

Let $\{\bar{x}_{p'}\}_{p'}$ be a Vitali subcollection, so that $\{B_{\bar{r}_{p'}/5}(\bar{x}_{p'})\}_{p'}$ are disjoint. By Theorem density-to-mass-bounds 5.4 at scale B_r, we deduce that

$$\sum_{p'} \bar{r}_{p'}^k \leq C_{mb}(\Lambda, \epsilon, n) r^k.$$

Therefore

$$\begin{aligned} \mu_i(B_r(x)) &\leq \sum_{p'} \mu_i(B_{\bar{r}_{p'}}(\bar{x}_{p'}) \\ &\leq \sum_{p'} c_2(n)\bar{r}_{p'}^k \\ &\leq c_2(n) C_{mb}(\Lambda, \epsilon, n) r^k. \end{aligned}$$

Ensuring $M \geq c_2 C_{mb}$ proves $(\dagger_i)$. □

This almost finishes proving Theorem 2.1 . We have shown the result provided the scale drop is sufficiently big, and with βr replacing r. Therefore, if $\{B_{r_{\alpha_i}}(x_i)\}_i$ is final the collection of balls from section 2, we break this up into $\sim 1/\rho$-many subcollections.

We can take ρ to be a power of 2. Then we define the subcollection U_β by

$$U_\beta = \{B_{r_{\alpha_i}}(x_i) : r_{\alpha_i} = \rho^\ell + r_\beta, \text{ for some integer } \ell\}.$$

For each U_β we have a packing bound by $C_{gmb}(n, \Lambda, \epsilon)$, therefore our final bound will be

$$C_{cover} = \frac{c(n) C_{gmb}}{\rho \beta^{n-k}},$$

which depends only on Λ, ϵ, n.