

NOTES ON A MEASURE THEORETIC VERSION OF NABER-VALTORTA'S RECTIFIABILITY THEOREM

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Here S typically represents the singular set, or more generally the (k, ϵ) -strata, of some geometric object. The monotone quantity typically quantifies how “homogenous” the geometric object is. Assumption 0.2 can usually be proven by a straightforward contradiction argument, using various compactness properties of the geometric object, (upper-semi-)continuity of the monotone quantity, and sharpness of the monotone quantity when it's constant in r . Assumption 0.3 usually follows from the “sharp” form of the monotone quantity. For examples, see [NV17] (harmonic maps), [NV15] (stationary varifolds), [NV16] (approximate harmonic maps), [EE17] (free-boundaries), [FS17] (thin obstacle problem).

We work in \mathbb{R}^n . Suppose there is a closed set S , and a non-negative quantity $\Theta(x, r)$, satisfying the following properties:

Assumption 0.1 (monotonicity). *For any $x \in B_1 \cap S$, and $0 < s < r < 1 - |x|$, we have*

$$\Theta(x, s) \leq \Theta(x, r) + \epsilon(r),$$

where $\epsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function satisfying

$$\int_0^2 \epsilon(r) \frac{dr}{r} < \infty.$$

Assumption 0.2 (quantitative splitting). *For any ρ, γ , there is an $\eta_0 = \eta_0(\gamma, \rho)$ so that the following occurs. Take $B_{10r}(x) \subset B_1$, and $\eta \leq \eta_0$. Suppose*

$$\epsilon(10r) \leq \eta, \quad \sup_{B_{3r}(x)} \Theta(z, 3r) \leq E.$$

Then at least one of the following occurs:

(1) *we have*

$$S \cap B_r(x) \subset \{z \in B_r(x) : \Theta(z, \gamma r) \geq E - \gamma\}, \quad \text{or}$$

(2) *there is an affine $k - 1$ -space $p + L^{k-1}$, so that*

$$\{z \in B_r(x) : \Theta(z, 3\eta r) \geq E - \eta\} \subset B_{\rho r}(p + L).$$

Assumption 0.3 (effective beta control). *There is a δ_0 so that the following occurs. Take $B_{10r}(x) \subset B_1$, and μ any finite Borel measure. Suppose*

$$\epsilon(10r) \leq \delta_0, \quad \Theta(x, 8r) - \Theta(x, \delta_0 r) < \delta_0, \quad x \in S.$$

Then

$$\beta_\mu^k(x, r)^2 \leq \frac{c_0}{r^k} \int_{B_r(x)} \Theta(z, 8r) - \Theta(r) + \epsilon(r) d\mu(z).$$

Here $\beta_\mu^k(x, r)$ is the Jones's β number, defined by

$$\beta_\mu^k(x, r)^2 = \inf_{p+V^k} r^{-k-2} \int_{B_r(x)} d(z, p+V)^2 d\mu(z),$$

where the infimum is over affine k -planes $p+V^k$.

Then it holds that

Theorem 0.4 ([NV17]). *Suppose $\sup_{x \in B_1} \Theta(x, 1) \leq \Lambda$, and $\theta \in (0, 1)$. There is a constant $c = c(n, \theta, \epsilon(r), c_0, \delta_0, \eta_0)$ so that $S \cap B_\theta$ is k -rectifiable, and*

$$\mathcal{H}^k(S \cap B_\theta) \leq c\Lambda^{1+c}.$$

In fact we have the estimate

$$\text{Vol}(B_r(S \cap B_\theta)) \leq c\Lambda^{1+c} r^{n-k} \quad \forall r < 1.$$

Remark 0.5. As observed by [NV17], a relatively minor modification of the proof would in fact prove the following. Given any collection of disjoint balls $\{B_{r_a}(a)\}_{a \in \mathcal{A}}$, with $\mathcal{A} \in S \cap B_{1/2}$ and $r_a \leq 1/10$, then there is a subcollection $\mathcal{A}' \subset \mathcal{A}$ satisfying:

$$B_{r_a}(\mathcal{A}) \subset B_{5r_a}(\mathcal{A}'), \quad \sum_{a' \in \mathcal{A}'} r_{a'}^k \leq c,$$

for $c = c(n, \epsilon(r), c_0, \delta_0, \eta_0)$.

We use the following discrete Reifenberg theorem.

Theorem 0.6 ([NV17] or [ENV16]). *Let $\{B_{r_p}(p)\}_p$ be a finite collection of disjoint balls, with $r_p \leq 1$. Let $\mu = \sum_p r_p^k \delta_p$. Suppose that*

$$\int_{B_1} \int_0^1 \beta_\mu^k(z, r)^2 dr/r d\mu(z) \leq 1.$$

Then

$$\mu(B_1) \leq c_{dr}(n).$$

And the following rectifiability theorem.

Theorem 0.7 ([AT15] or [ENV16]). *Let $\mu = \mathcal{H}^k \llcorner S$, for $S \subset B_1$. Suppose*

$$\int_0^1 \beta_\mu^k(z, r)^2 dr/r < \infty \quad \mu - a.e.z.$$

Then S is k -rectifiable.

Lemma 0.8. *There is an $\eta_1(n, \delta_0, c_0)$ so that the following holds. Take $B_{2r}(q) \subset B_1$. Take r sufficiently small so that $\int_0^r \epsilon dr/r < \eta_1$. Let $\{B_{r_p}(p)\}_p$ be a finite collection of disjoint balls, satisfying*

$$\Theta(p, r) - \Theta(p, \delta_0 r_p) \leq \eta_1, \quad p \in S \cap B_r(q), \quad r_p \leq r.$$

Then we have

$$\sum_p r_p^k \leq c_1(n)r^k.$$

Proof. Choose η_1 to be the lesser of

$$\delta_0/5, \quad \frac{1}{16c_0 20^n (1 + c_{dr})^2}.$$

Write $r_i = 2^{-i}r$. We assume $i \geq 5$, so that $r_i \leq r/32$. Define the packing measure

$$(1) \quad \mu_i = \sum_{r_p \leq r_i} r_p^k \delta_p,$$

and for shorthand write $\beta_i^k = \beta_{\mu_i, 2}^k$.

We make a few remarks about the β_i . Suppose $x \in \text{spt} \mu_i$, and $j \geq i$. Then by disjointness

$$(2) \quad \beta_i(x, r_j) = \begin{cases} \beta_j(x, r_j) & \text{if } x \in \text{spt} \mu_j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\epsilon(10r_i) < 2\eta_1$, and $\Theta(p, 8r_i) - \Theta(\delta_0 r_i) \leq 5\eta_1$, and $p \in S$, we have by our assumption 0.3 that

$$(3) \quad \beta_i(x, r_i)^2 \leq c_0 r_i^{-k} \int_{B_{r_i}(x)} \Theta(y, 8r_i) - \Theta(y, r_i) + \epsilon(8r_i) d\mu_i(y).$$

We prove inductively that, for $i \geq 5$, we have the estimate

$$(4) \quad \mu_i(B_{r_i}(x)) \leq c_{dr} r_i^k,$$

where c_{dr} is the constant from discrete Reifenberg. The required estimate will then follow by a obvious packing argument.

(4) holds true for i sufficiently large, so that $r_i \leq \min_p r_p$. By inductive hypothesis, let us assume (4) holds at $i - 1$. By a straightforward packing estimate, we can estimate

$$\mu_j(B_{2r_j}(x)) \leq \Gamma r^k \quad \forall x \in \text{spt} \mu_j, \forall j \geq i,$$

where $\Gamma(n) = 20^n(1 + c_{dr}(n))$.

Then we have

$$\begin{aligned}
 & \sum_{r_j \leq r_i} \int_{B_{r_i}(x)} \beta_i(z, r_j)^2 d\mu_i(z) \\
 &= \sum_{r_j \leq r_i} \int_{B_{r_i}(x)} \beta_j(z, r_j)^2 d\mu_j(z) \\
 &\leq c_0 \sum_{r_j \leq r_i} \int_{B_{r_i}(x)} \int_{B_{r_j}(z)} \Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j) d\mu_j(y) d\mu_j(z) \\
 &\leq c_0 \int_{B_{2r_i}(x)} \sum_{r_j \leq r_i} \frac{\mu_j(B_{r_j}(y))}{r_j^k} (\Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j)) d\mu_j(y) \\
 &\leq c_0 \Gamma \int_{B_{2r_i}(x)} \sum_{r_y \leq r_j \leq r_i} \Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j) d\mu_j(y) \\
 &\leq 4c_0 \Gamma^2 r_i^k \sup_{y \in \text{spt}\mu_i \cap B_{2r_i}(x)} \left(|\Theta(y, 8r_i) - \Theta(y, r_y)| + \epsilon(8r_i) + \int_{r_y}^{16r_i} \epsilon dr/r \right) \\
 &\leq 8c_0 \Gamma^2 r_i^k \sup_p \left(|\Theta(p, r) - \Theta(p, \eta_1 r_p)| + \int_0^r \epsilon dr/r \right) \\
 &\leq 16c_0 \Gamma^2 \eta r_i^k \\
 &\leq r_i^k.
 \end{aligned}$$

Computations: For $y \in \text{spt}\mu_i$, we have

$$\sum_{r_y \leq r_j \leq r_i} \epsilon(8r_j) \leq \frac{1}{\log 2} \int_{r_y}^{16r_i} \epsilon dr/r.$$

And

$$\sum_{r_y \leq r_j \leq r_i} \Theta(y, 8r_j) - \Theta(y, r_j) \leq 2(\Theta(y, 8r_i) - \Theta(y, r_y) + \epsilon(8r_i)).$$

Then discrete Reifenberg implies that

$$\mu_i(B_{r_i}(x)) \leq c_{dr}(n) r_i^k.$$

This proves the inductive claim. \square

Theorem 0.9. *For any ρ , there is an $\eta_2(n, c_0, \delta_0, \eta_0, \rho)$ so that the following holds. Take any $\eta \leq \eta_2$, and take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{10r} \epsilon dr/dr < \eta/16$, and fix $R > 0$. Suppose $\sup_{x \in B_{3r}(p)} \Theta(p, 3r) \leq E$, and take $p' \in B_r(p)$, $r' \leq r$.*

There is a collection of balls $\{B_{r_s}(s)\}_{s \in \mathcal{S}} \cup \{B_{r_b}(b)\}_{b \in \mathcal{B}}$ satisfying:

- (1) $\mathcal{B} \cup \mathcal{S} \subset S \cap B_{r'}(p') \cap B_r(p)$ and $r_s \leq r'$, $r_b \leq r'$,
- (2) $S \cap B_{r'}(p') \subset B_{r_s}(\mathcal{S}) \cup B_{r_b}(\mathcal{B})$,
- (3) $\sum_{s \in \mathcal{S}} r_s^k + \sum_{b \in \mathcal{B}} r_b^k \leq c_2(n) r'^k$,
- (4) For every $s \in \mathcal{S}$, we have $r_s = \min\{r', R\}$, and for every $b \in \mathcal{B}$, we have

$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta r_b) \geq E - \eta/2\} \subset B_{\rho r_b}(q + L^{k-1})$$

for some $(k-1)$ -plane $q + L^{k-1}$.

Proof. Choose

$$(5) \quad \gamma = \min\{\delta_0, \eta_1(n, \delta_0, c_0)/2\}$$

$$(6) \quad \eta_2 = \min\{\gamma/10, \eta_0(\gamma, \rho)/10\}.$$

If $R \geq r'$, just let $\mathcal{S} = \{p'\}$, and $r_{p'} \equiv r_s = r'$. Otherwise proceed as follows. By monotonicity, for any ball $x \in B_r(p)$, and $s \leq r$, we have $\Theta(x, s) \leq E + \eta/2$. Then we have by assumption 0.2 that one of the following occurs:

$$\Theta(z, \gamma s) \geq E - \gamma \text{ on } S \cap B_s(x),$$

or

$$\{z \in B_s(x) : \Theta(z, 3\eta s) \geq E - \eta/2\} \subset B_{\rho s}(q + L^{k-1}),$$

for some $(k-1)$ -space $q + L^{k-1}$.

For every $x \in B_{r'}(p') \cap B_r(p)$, let

$$r_x = \inf\{r'\} \cup \{s \geq R : \Theta(z, \gamma s) \geq E - \gamma \text{ on } S \cap B_s(x)\}.$$

Trivially $r_x \leq r'$. Then $\{B_{r_x}(x)\}_{x \in S \cap B_{r'}(p') \cap B_r(p)}$ covers $S \cap B_{r'}(p) \cap B_r(p)$. Let $\{B_{r_x}(x)\}_{x \in \mathcal{U}}$ be a Vitali cover, so that the $r_x/5$ -balls are disjoint. For every $x \in \mathcal{U}$, we have

$$\Theta(x, r') - \Theta(x, \delta_0 r_x/5) \leq \gamma + \eta/4 \leq \eta_1(n, \delta_0, c_0),$$

and therefore by Theorem 0.6, we have the required packing estimate with $c_2 = 5^k c_1$. Set $\mathcal{S} = \{x \in \mathcal{U} : r_x = R\}$, and $\mathcal{B} = \{x \in \mathcal{U} : r_x > R\}$, and the required properties follow by assumption 0.2. \square

Theorem 0.10. *There is a $\rho(n)$, $\gamma(n, c_0, \delta_0)$, $\eta_3(n, c_0, \delta_0, \eta_0(\gamma, \rho))$ so that the following holds. Take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{16r} \epsilon dr/r < \eta_3/16$. Suppose $\sup_{y \in B_{3r}(p)} \Theta(y, 3r) \leq E$. Take any $0 < R \leq r$.*

There is a collection of balls $\{B_{r_y}(y)\}_{y \in \mathcal{U}}$ which satisfy the following properties:

- (1) $\mathcal{U} \subset S \cap B_r(p)$, and $r_y \leq \max\{r/10, R\}$,
- (2) $S \cap B_r(p) \subset B_{r_y}(\mathcal{U})$,
- (3) $\sum_{y \in \mathcal{U}} r_y^k \leq c_6 r^k$,
- (4) for every $y \in \mathcal{U}$, we have

$$\text{either } r_y = R \quad \text{or} \quad \sup_{z \in B_{3r_y}(y)} \Theta(y, 3r_y) \leq E - \eta_3/2.$$

Here $c_6 = c_6(n, c_0, \delta_0, \eta_0(\gamma, \rho))$.

Proof. In the following we will fix $\rho = \rho(n)$. Let $\gamma(n, c_0, \delta_0)$ be as in Lemma 0.8, and let $\eta_3 = \min\{\eta_2, \rho\}$.

If $R \geq r/10$, then simply take $\{B_{r_y}(y)\}_{y \in \mathcal{U}}$ to be a Vitali subcover of $\{B_R(y)\}_{y \in S \cap B_r(p)}$, so that the $R/5$ -balls are disjoint. Otherwise, we proceed as follows. We build a sequence of covering $\mathcal{S}_i, \mathcal{B}_i$ ($i = 0, 1, 2, \dots$) satisfying the following inductive hypotheses:

- (1) size control

$$\mathcal{S}_i \cup \mathcal{B}_i \subset B_r(p), \quad \eta_3 R \leq r_s \leq r/10, \quad \eta_3 R \leq r_b \leq 2^{-i} r.$$

- (2) bad ball packing:

$$\sum_{b \in \mathcal{B}_i} r_b^k \leq c_2(n) r 2^{-i}.$$

- (3) stop ball packing:

$$\sum_{s \in \mathcal{S}_i} r_s^k \leq c_3(n, \eta_3) r \sum_{j=-1}^{i-1} 2^{-j}.$$

- (4) covering:

$$S \cap B_r(p) \subset B_{r_s}(\mathcal{S}) \cup B_{r_b}(\mathcal{B}).$$

- (5) stop ball control: for every $s \in \mathcal{S}_i$, we have

$$r_s \in [\eta_3 R, R] \quad \text{or} \quad \sup_{y \in B_{3r_s}(s)} \Theta(y, 3r_s) \leq E - \eta_2/2.$$

- (6) bad ball control: for every $b \in \mathcal{B}_i$, we have we have

$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta_3 r_b) \geq E - \eta_3/2\} \subset B_{\rho r_b}(q + L^{k-1})$$

for some $(k-1)$ -plane $q + L^{k-1}$.

Apply Theorem 0.9 with $r' = r$, $p' = p$ to obtain a $\mathcal{S}_0, \mathcal{B}_0$. These trivially satisfy the required estimates with c_2 in place of c_3 . Provided we take $c_3 \geq c_2$, this starts our inductive procedure.

Let us assume we have constructed $\mathcal{S}_i, \mathcal{B}_i$ satisfying inductive hypotheses. Fix a $b \in \mathcal{B}_i$. By bad ball control, there is an affine $(k-1)$ -space $q + L^{k-1}$ so that

$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta_3 r_b) \geq E - \eta_3/2\} \subset B_{\rho r_b}(q + L).$$

Let \mathcal{J}_b be a maximal ρr_b -net in $B_{10\rho r_b}(q + L) \cap B_{r_b}(b) \cap B_r(p)$. Then

$$\#\mathcal{J}_b \leq c_4(n)\rho^{1-k},$$

and

$$B_{10\rho r_b}(q + L) \cap B_{r_b}(b) \cap B_r(p) \subset \cup_{z \in \mathcal{J}_b} B_{\rho r_b}(z).$$

For each $z \in \mathcal{J}_b$, apply Theorem 0.9 with $r' = \rho r_b$, $p' = z$, to obtain a collection $\mathcal{B}_{b,z}, \mathcal{S}_{b,z}$. Then

$$S \cap B_{r_b}(b) \cap B_r(p) \subset B_{r_s}(\mathcal{S}_{b,z}) \cup B_{r_b}(\mathcal{B}_{b,z}),$$

and $r_s = R$ for all $s \in \mathcal{S}_{b,z}$. Define

$$\mathcal{B}_{i+1} = \bigcup_{b \in \mathcal{B}_i} \bigcup_{z \in \mathcal{J}_b} \mathcal{B}_{b,z}.$$

We have

$$(7) \quad \sum_{b \in \mathcal{B}_{i+1}} r_b^k \leq \sum_{b \in \mathcal{B}_i} \sum_{z \in \mathcal{J}_b} c_2(\rho r_b)^k$$

$$(8) \quad \leq \sum_{b \in \mathcal{B}_i} c_2 c_4 \rho r_b^k$$

$$(9) \quad \leq c_2^2 c_4 \rho 2^{-i} r$$

$$(10) \quad \leq c_2 2^{-i-1} r,$$

provided $\rho = \rho(n)$ is chosen sufficiently small so that $c_2(n)c_4(n)\rho \leq 1/2$ and $\rho \leq 1/10$. This constructs the required \mathcal{B}_{i+1} .

We build the required \mathcal{S}_{i+1} . For each $b \in \mathcal{B}_i$, let \mathcal{S}_b be a maximal $\eta_3 r_b$ -net in $B_{r_b}(b) \cap B_r(p) \setminus B_{10\rho r_b}(q + L^{k-1})$. For every $s \in \mathcal{S}_b$, let $r_s = \eta_3 r_b$. Then

$$\sum_{s \in \mathcal{S}_b} r_s^k \leq c_5(n, \eta_3) r_b^k,$$

and

$$B_{r_b}(b) \cap B_r(p) \setminus B_{10\rho r_b}(q + L) \subset \cup_{s \in \mathcal{S}_b} B_{r_s}(s).$$

Moreover, since $B_{3r_s}(s) \cap B_{\rho r_b}(q + L) = \emptyset$, we have

$$\sup_{y \in B_{3r_s}(s)} \Theta(y, 3r_s) \leq E - \eta_3/2.$$

Let

$$\mathcal{S}_{i+1} = \mathcal{S}_i \cup \bigcup_{b \in \mathcal{B}_i} \left(\mathcal{S}_b \cup \bigcup_{z \in \mathcal{J}_z} \mathcal{S}_{z,b} \right).$$

Then we have

$$(11) \quad \sum_{s \in \mathcal{S}_{i+1}} r_s^k \leq c_3 r \sum_{j=-1}^{i-1} 2^{-j} + \sum_{b \in \mathcal{B}_i} \left(\sum_{s \in \mathcal{S}_b} r_s^k + \sum_{z \in \mathcal{J}_z} \sum_{s \in \mathcal{S}_{b,z}} r_s^k \right)$$

$$(12) \quad \leq c_3 r \sum_{j=-1}^{i-1} 2^{-j} + c_5 \sum_{b \in \mathcal{B}_i} r_b^k + 2^{-1} \sum_{b \in \mathcal{B}_i} r_b^k$$

$$(13) \quad \leq c_3 r \sum_{j=-1}^{i-1} 2^{-j} + (c_5 + 1) c_2 2^{-i} r$$

$$(14) \quad \leq c_3 r \sum_{j=-1}^i 2^{-j},$$

provided $c_3 \geq (c_5 + 1) c_2$.

This proves the required estimates, and finishes the construction of the required $\mathcal{B}_i, \mathcal{S}_i$.

Notice that for i sufficiently large, we have $\mathcal{B}_i = \emptyset$, and hence $B_{r_s}(\mathcal{S}_i)$ covers $S \cap B_r(p)$. For this i , let $\mathcal{U} = \mathcal{S}_i$, and $r_x = \max\{R, r_s\}$. Then \mathcal{U} satisfies the required conditions, with $c_6 = \eta_3^{-k} c_3 4$. \square

Theorem 0.11. *There is an $\eta_4(n, c_0, \delta_0, \eta_0)$ so that the following holds. Take $B_{11r}(p) \subset B_1$, with r chosen sufficiently small so that $\int_0^{10r} \epsilon dr/r < \eta_4$. Suppose $\sup_{y \in B_{3r}(p)} \Theta(y, 3r) \leq E$.*

Then for any $s \leq r$, there is a subset $\mathcal{U} \subset S \cap B_r(p)$, so that

$$S \cap B_r(p) \subset \cup_{y \in \mathcal{U}} B_s(y),$$

and

$$\#\mathcal{U} \leq c_7^{1+2E/\eta_3} s^{-k},$$

where $c_7 = c_7(n, c_0, \delta_0, \eta_0)$.

Proof. Fix $\eta_4 = \eta_3/16$. There is no loss in replacing S with $S \cap B_r(p)$. We construct a sequence of coverings \mathcal{U}_i ($i = 1, 2, \dots$) which satisfy the following

(1) covering:

$$S \cap B_r(p) \subset B_{r_x}(\mathcal{U}_i),$$

(2) packing:

$$\sum_{x \in \mathcal{U}_i} r_x^k \leq c_8 \sum_{x \in \mathcal{U}_{i-1}} r_x^k, \quad \sum_{x \in \mathcal{U}_1} r_x^k \leq c_8 r^k,$$

for $c_8 = c_8(n, c_0, \delta_0, \eta_0)$,

(3) energy drop: for every $x \in \mathcal{U}_i$, we have $r_x \geq s$, and either

$$r_x = s \quad \text{or} \quad \sup_{y \in B_{3r_x}(x)} \Theta(y, 3r_x) \leq E - i\eta_3/2.$$

(4) radius control:

$$\sup_{x \in \mathcal{U}_i} r_x \leq \max\{s, 10^{-i}r\}.$$

Apply Theorem 0.10 with $R = s$ to obtain \mathcal{U}_1 . This starts our inductive process. Suppose we have constructed \mathcal{U}_i . We demonstrate how to build \mathcal{U}_{i+1} . Fix $x \in \mathcal{U}_i$. If $r_x = s$, then let $\mathcal{U}'_x = \{x\}$. Otherwise, apply Theorem 0.10 with $R = s$, $p = x$, $r = r_x$, $E - i\eta_3/2$ in place of E , to obtain a family \mathcal{U}'_x . Define

$$\mathcal{U}_{i+1} = \bigcup_{x \in \mathcal{U}_i} \mathcal{U}'_x.$$

Observe that for $i \geq 1+2E/\eta_3$, we must have $r_x = s$ for every $x \in \mathcal{U}_i$, since Θ is a non-negative function. We deduce that this \mathcal{U}_i gives the required covering. \square

Proof of Theorem 0.4. Choose r_0 sufficiently small so that $11r_0 \leq 1 - \theta$, $\int_0^{10r_0} \epsilon dr/r \leq \eta_4$. Let $\{B_{r_0}(x)\}_{x \in \mathcal{U}'}$ be a Vitali cover of $S \cap B_\theta$, so that the $r_0/5$ -balls are disjoint. For each $x \in \mathcal{U}'$, apply Theorem 0.11 with $s = r$ and $E = \Lambda + \epsilon(1)$ to obtain a \mathcal{U}_x . Define the cover $\mathcal{U} = \cup_{x \in \mathcal{U}'} \mathcal{U}_x$. Then we have

$$S \cap B_\theta \subset \bigcup_{x \in \mathcal{U}} B_r(x),$$

and

$$\#\mathcal{U} \leq c^{1+c(\Lambda+\epsilon(1))} \#\mathcal{U}' \leq c^{1+c\Lambda},$$

where $c = c(n, \theta, \epsilon, c_0, \delta_0, \eta_0)$. This proves the required Minkowski estimate, and hence the required Hausdorff estimate also.

To prove rectifiability, one uses a minor modification of Theorems 0.9-0.11 with $R = 0$, to break into covers of the form $\mathcal{U}_0 \cup B_{r_y}(\mathcal{U})$, where \mathcal{U}_0 is rectifiable, and each $B_{r_y}(y)$ admits a small but definite density drop. We state the theorems below, but leave most of the proofs as a straightforward exercise for the reader. \square

In the following, we make the following additional assumption on S , which we just showed follows from Assumptions 0.1, 0.2, 0.3.

Assumption 0.12. *For any $\theta < 1$, there is some constant $\Gamma(\theta)$ so that*

$$\mathcal{H}^k(S \cap B_r(x)) \leq \Gamma r^k \quad \forall x \in B_\theta, r < 1 - |x|,$$

Lemma 0.13 (lemma 0.8'). *Take $B_{11r}(q) \subset B_1$. Take r sufficiently small so that $\epsilon(10r) < \delta_0/16$. Suppose \mathcal{S} is a subset of $S \cap B_r(q)$, satisfying*

$$\Theta(p, 8r) - \Theta(p, 0) \leq \delta_0/16 \quad \forall p \in \mathcal{S}, s \leq r.$$

Then \mathcal{S} is rectifiable.

Proof. Define the Radon measure $\mu = \mathcal{H}^k \llcorner \mathcal{S}$. Let $r_i = 2^{-i}r$. I claim that for any $i \geq 5$, and any $x \in B_r(p)$, we have

$$\sum_{r_j \leq r_i} \int_{B_{r_j}(x)} \beta_\mu^k(z, r_j)^2 d\mu(z) < \infty.$$

Rectifiability will follow from Theorem 0.7.

For every $x \in \mathcal{S}$, we have $\epsilon(10r) < \delta_0$, and $\Theta(x, 8r_i) - \Theta(x, r_i) < \delta_0/16 + \delta_0/8 < \delta_0$, and therefore by assumption 0.3

$$\beta_\mu^k(x, r_i)^2 \leq c_0 r_i^{-k} \int_{B_{r_i}(x)} \Theta(y, 8r_i) - \Theta(y, r_i) + \epsilon(8r_i) d\mu(y).$$

We compute:

$$\begin{aligned} & \sum_{r_j \leq r_i} \int_{B_{r_j}(x)} \beta_\mu^k(z, r_j)^2 d\mu(z) \\ & \leq c_0 \sum_{r_j \leq r_i} \int_{B_{r_j}(x)} \int_{B_{r_j}(z)} \Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j) d\mu(y) d\mu(z) \\ & \leq c_0 \Gamma \int_{B_{2r_i}(x)} \sum_{r_j \leq r_i} \Theta(y, 8r_j) - \Theta(y, r_j) + \epsilon(8r_j) d\mu(y) \\ & \leq c_0 \Gamma^2 \left(\delta_0 + \int_0^{16r_i} \epsilon dr/r \right) \\ & < \infty. \end{aligned} \quad \square$$

Theorem 0.14 (theorem 0.9'). *For any ρ , there is an $\eta_2(n, c_0, \delta_0, \eta_0, \rho)$ so that the following holds. Take any $\eta \leq \eta_5$, and take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{10r} \epsilon dr/r < \eta/16$. Suppose $\sup_{x \in B_{3r}(p)} \Theta(p, 3r) \leq E$, and take $p' \in B_r(p)$, $r' \leq r$.*

There is a set \mathcal{S}_0 , and a collection of balls $\{B_{r_b}(b)\}_{b \in \mathcal{B}}$ satisfying:

- (1) $\mathcal{S}_0 \cup \mathcal{B} \subset S \cap B_{r'}(p') \cap B_r(p)$, and $r_b \leq r'$,

- (2) $S \cap B_{r'}(p') \subset \mathcal{S}_0 \cup B_{r_b}(\mathcal{B})$,
- (3) $\sum_{b \in \mathcal{B}} r_b^k \leq c_2(n)r'^k$,
- (4) \mathcal{S} is k -rectifiable, and for every $b \in \mathcal{B}$, we have

$$\{z \in B_{r_b}(b) : \Theta(z, 3\eta r_b) \geq E - \eta/2\} \subset B_{\rho r_b}(q + L^{k-1}),$$
 for some $(k-1)$ -plane $q + L^{k-1}$.

Proof. Same as Theorem 0.9, except we let $R = 0$, and then define $\mathcal{S}_0 = \{x \in \mathcal{U} : r_x = 0\}$, and use Lemma 0.8' to prove rectifiability (in addition to Lemma 0.8 to prove packing control on the bad balls). \square

Theorem 0.15 (theorem 0.10'). *There is a $\rho(n)$, $\gamma(n, c_0, \delta_0)$, $\eta_3(n, c_0, \delta_0, \eta_0(\gamma, \rho))$ so that the following holds. Take $B_{11r}(p) \subset B_1$, with r sufficiently small so that $\int_0^{16r} \epsilon dr/r < \eta_3/16$. Suppose $\sup_{y \in B_{3r}(p)} \Theta(y, 3r) \leq E$.*

There is a set \mathcal{U}_0 , and a collection of balls $\{B_{r_y}(y)\}_{y \in \mathcal{U}}$, which satisfy the following properties:

- (1) $\mathcal{U}_0 \cup \mathcal{U} \subset S \cap B_r(p)$, and $r_y \leq r/10$,
- (2) $S \cap B_r(p) \subset \mathcal{U}_0 \cup B_{r_y}(\mathcal{U})$,
- (3) $\sum_{y \in \mathcal{U}} r_y^k \leq c_6 r^k$,
- (4) \mathcal{U}_0 is k -rectifiable, and for every $y \in \mathcal{U}$, we have

$$\sup_{z \in B_{3r_y}(y)} \Theta(y, 3r_y) \leq E - \eta_3/2.$$

Proof. Same as Theorem 0.10, except with $R = 0$, and in the inductive construction, use that at stage i we have $\mathcal{H}^k(S \setminus \mathcal{S}_0 \cup B_{r_s}(\mathcal{S})) \leq c_2 r 2^{-i} \rightarrow 0$. \square

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