Notes from Brian White's class on mean curvature flow

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1 evolution of curves

We consider the evolution of an embedded curve $\Gamma \subset R^2$ by its curvature vector. In other words, a 1-parameter family of embedded curves

$$t \in I \mapsto \Gamma_t$$

such that $\partial_t \Gamma = \mathbf{k}$.

Curve shortening flow has the following general properties:

- smoothing: if initially C^2, then becomes instantly analytic
- singularities occur, in finite time
- circle of radius r_0 shrinks through circles of radius $r(t) = \sqrt{r_0^2 - 2t}$
- arclength descreases
- avoidance: disjoint curves stay disjoint
- embeddedness is preserved

In the next several sections we work towards proving Grayon's theorem, classifying the long-time behavior of curve shortening flow in manifolds. The proof we present is not the most direct, but presents ideas we apply later to study higher dimensional mean curvature flows.

Theorem 1.1. *We have*

$$(\textit{area enclosed})'(t) = -2\pi.$$

In particular,

$$\textit{lifetime} \leq \frac{\textit{initial area}}{2\pi}.$$

Proof. Let ν be the outer normal along Γ. We have

$$\begin{aligned}
(\text{area enclosed})'(t) &= \int_{\Gamma_t} \partial_t \cdot \nu \\
&= \int \mathbf{k} \cdot \nu \\
&= -2\pi,
\end{aligned}$$

by Gauss-Bonnet. □

The following two Theorems completely describe the behavior of curve-shortening flow.

Theorem 1.2 (Gage-Hamilton)**.** *Any convex curve in R^2 shrinks to a round point.*

Remark 1.3. The "2/3-power" flow

$$\partial_t = \frac{1}{|k|^{2/3}}\mathbf{k}$$

has the property that ellipses remain elliptical, with the same eccentricity. In particular, convex curves will collapse to an elliptical point. In fact this flow is affine invariant, in the sense that the image of a self-shrinker under an linear map will also be a self-shrinker.

Here's an interesting question: given the initial convex curve, can you predict what kind of ellipse you obtain in the limit?

One can consider more general "r-power" flows

$$\partial_t = \frac{1}{|k|^{r}}\mathbf{k}.$$

If $r > 2/3$ than you can shrink to a point in the original flow, but in the normalized flow (c.f. next section) you're length will go to infinity. Essentially your ellipticity goes to infinity.

This is probably true: suppose you start with a convex curve under the r-power flow. Then for each $r \geq 0$ you will shrink to a point but in the normalized flow the following will happen. If $r < 2/3$ the ellipticity goes to 0. If $r = 2/3$ ellipticity is preserved. If $r > 2/3$ the ellipticity goes to infinity.

Theorem 1.4 (Grayson)**.** *Any simple closed curve in R^2 becomes convex, and consequently shrinks to a round point.*

Corollary 1.5. *We have equality*

$$\textit{lifetime} = \frac{\textit{enclosed area}}{2\pi}.$$

For curve-shortening flow in surfaces a similar Theorem is known.

Theorem 1.6 (Grayson)**.** *If $\Gamma \subset M$, where M is a compact surface, then either Γ_t shrinks to a round point in finite time, or a closed geodesic in infinite time.*

One can ask if, given the initial Γ, whether you can predict the location of the final point. Bryant answered this in the negative, by showing that the only "conserved quantity" is $A' = -2\pi$. Here's another question: can you predict whether you'll shrink to a point or a geodesic?

Corollary 1.7 (LusternikSchnirelmann)**.** *Every metric on S^2 has ≥ 3 closed geodesic.*

Proof. (sketch) In the standard metric, consider the family of $\{C_s\}_{s\in(-1,1)}$ circles obtain by slicing (S^2, δ) by parallel hyperplanes. Orient all the C_s in the same direction. Consider the curve shortening flow of each C_s in our metric g.

Observe that for s very near -1, the flow $C_s(t)$ becomes a round point in finite time. Similarly, for s very near 1, the flow also shrinks $C_s(t)$ to a round point, but with opposite orientation. The family C_s is smooth away from $\{-1,1\}$, and curve shortening flow (of compact curves) depends smoothly on the initial conditions. Therefore at some $s \in (-1,1)$ the flow cannot shrink to a round point, and consequently approaches a geodesic.

To obtain the second and third geodesic, consider the 2- and 3-parameter families of curves by rotating the C_s through one and two axes, then play a similar game...

□

2 max principles

Theorem 2.1. *Let M be a compact manifold, and $f : M \times [0,T) \to R$. Let $\phi(t) = \min_M f(\cdot,t)$, and suppose that whenever $\phi(t) = f(x,t)$, we have $\partial_t f|_{(x,t)} \geq 0$. Then ϕ is increasing in t.*

Proof. Suppose that actually $\partial_t f > 0$ at any minimum. Then trivially $\min_{[a,T)} \phi(t)$ must be realized at the a. For more general f, apply the above reasoning to $f + \epsilon t$, and let $\epsilon \to 0$. □

Theorem 2.2. *Let $\Gamma_1(t)$ and $\Gamma_2(t)$ be compact, embedded, disjoint curves moving by MCF. Then $\mathrm{dist}(\Gamma_1, \Gamma_2)$ is increasing in time.*

Proof. Parameterize Γ_i by $F_i : S^1 \times [0,T) \to R^2$. Let

$$\begin{aligned} f &: S^1 \times S^1 \times [0,T) \to R \\ &: (\theta_1, \theta_2, t) \mapsto |F_1(\theta_1,t) - F_2(\theta_2,t)|^2. \end{aligned}$$

At any minimum (θ_1,θ_2) of $f(\cdot,\cdot,t)$, we have

$$\partial_t f = 2(F_1 - F_2)\cdot(\mathbf{k_1} - \mathbf{k_2}) \geq 0.$$

To see this, observe that necessarily $\nu_1, \nu_2, F_1 - F_2$ are all parallel. Now translate Γ_1 and Γ_2 along $F_1 - F_2$ until they touch. Since they touch to first order, and locally lie to one side of each other, we have

$$\mathbf{k_1}\cdot(F_1 - F_2) \geq \mathbf{k_2}\cdot(F_1 - F_2).$$ □

Theorem 2.3. *Let Ω be the upper half plane in R^2, and $\Gamma(t)$ a MCF of closed embedded curves. Then # connected components of $\Gamma(t) \cap \Omega$ is decreasing in time.*

Proof. Let U be a connected component of $\{(\theta,t) : F_t(\theta)\cdot e_2 \geq 0\}$. By analyticity $F \cdot e_2 > 0$ in $intU \cap \{t > 0\}$, and $F \cdot e_2 = 0$ on $\partial U \cap \{t > 0\}$.

At any maximum of $F_t \cdot e_2$ in U, with $t > 0$, we have $\partial_t F_t \cdot e_2 \leq 0$. So by the maximum principle we must have that $\max_U F_t \cdot e_2$ is decreasing in time. We deduce there is a point $(\theta, 0) \in intU$.

Since the same argument applies to the lower-half plane, the number of components of U_t is decreasing in time. □

Corollary 2.4. *If Γ_t is the MCF of closed, embedded curve, and L is a line, then $\#\Gamma(t) \cap L$ is decreasing in time.*

Here's a question: is it possible to flow with boundary, and move the boundary so that at a finite time Γ_t agrees with a line?

3 unparameterized MCF

The equation *velocity* $= H$ specifies not just a flow but a parameterization: the velocity is precisely normal to the surface, and equal to H. It is sometimes convenient to work with different kinds of parameterization for the same flow, and more generally think of MCF as a purely geometric object. Geometrically the motion of MCF is described by

$$vel^{\perp} = H.$$

If M_t is a geometric, n-dimensional MCF in R^{n+1}, then we can locally (in spacetime) write M_t as the evolution of a graph. This allows us to reduce the problem to a parabolic PDE.

Theorem 3.1. *Let $u : \Omega \times I \to R$, for $\Omega \subset R^n$, and I an interval. Then* graph$u$ *moves by MCF iff*

$$\begin{aligned}\partial_t u &= \sqrt{1+|Du|^2} D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right) \\ &= \Delta u - \frac{D_i u D_j u}{1+|Du|^2} D_i D_j u.\end{aligned}$$

Proof. Write $x = (x', x_{n+1}) \in R^{n+1} = R^n \times R$. Of course the velocity of graph$u$ is $(0, \partial_t u)$. We have

$$H = H \cdot n = D_i \left(\frac{D_i u}{\sqrt{1+|Du|^2}} \right).$$

To move by (geometric) MCF, we require

$$H = vel \cdot n = (0, \partial_t u) \cdot \frac{(-Du, 1)}{\sqrt{1+|Du|^2}}.$$ □

4 long term behavior

We analyze the long-time behavior of MCF of curves.

Theorem 4.1. *Let $t \in [0, \infty) \mapsto \Gamma(t)$ be a MCF of closed, embedded curves in a compact manifold N. Let $\Gamma_i(t) = \Gamma(t - t_i)$, for some sequence $t_i \to \infty$. Then there is a subsequence i' such that*

$$\Gamma_{i'}(\cdot) \to \textit{closed geodesic},$$

where the convergence is C^∞ on compact subsets of spacetime.

Remark 4.2. Different choices of sequence could potentially give different geodesics. This is strongly related to non-uniquess of tangent maps. For example, let $f : R^2 \to R$ be a positive C^∞ function, with the property that some integral curve $\gamma(t)$ of ∇f takes infinitely long to spiral into a circle. E.g.

$$f(r\theta) = 2 - \exp(-1/r^2)\sin(\theta + 1/r).$$

Let $N = S^1 \times R^2$, with the metric $f(x,y)ds^2 + dx^2 + dy^2$. Starting at an appropriate $S^1 \times \{x_0, y_0\}$, then curve shortening flow is precisely this integral curve of ∇f. By choosing different sequences you can obtain any $S^1 \times \{x, y\}$, where (x, y) lie on the circle we approach.

As suggested by this example, the set of limit geodesics is always connected. Further if N is analytic the limiting geodesic should be unique.

Proof. Since $L\Gamma(t)$ is decreasing in time, we have for any $a < b$,

$$\begin{aligned}
&L\Gamma(t_i + a) - L\Gamma(t_i + b) \to 0 \quad \text{as } i \to \infty \\
&= \int_{t_i+a}^{t_i+b} k^2 ds \\
&= \int_a^b \int_{\Gamma_i(t)} k^s ds dt.
\end{aligned}$$

After passing to a subsequence we have

$$\sum_i \int_a^b \int_{\Gamma_i(t)} k^2 ds dt < \infty.$$

Hence

$$\int_a^b \sum_i \int_{\Gamma_i(t)} k^2 ds dt < \infty,$$

and therefore

$$\sum_i \int_{\Gamma_i(t)} k^2 ds < \infty \quad \text{for a.e. } t \in [a, b].$$

Thus, for a.e. $t \in [a, b]$,

$$\int_{\Gamma_i(t)} k^2 ds \to 0 \quad \text{as } i \to \infty.$$

Fix such a time T. Parameterizing each $\Gamma_i(T)$ by arclength, the above says that $\{\Gamma_i(T)\}$ is uniformly bounded in $W^{2,2}(S^1)$. After passing to a subsequence we have weak convergence $\Gamma_i(T) \to C$ in $W^{2,2}(S^1) \subset C^{1,\alpha}(S^1)$. Therefore C is parameterized by arclength, with weak second derivatives being zero. Thus C is a geodesic.

Since $F_i(T) \to C$ in $C^{1,\alpha}$, and C is compact, we have that $\Gamma_i(\cdot) \to C$ smoothly on compact subsets of $S^1 \times (0, \infty)$, by smooth depends on initial conditions. $\square$

Remark 4.3 (higher dimensions)**.** Let $M(t)$ be a MCF of m-surfaces in $R^N \times [0,\infty)$, and $t_i \to \infty$. Set $M_i(t) = M(t - t_i)$. Then the above proof shows that for a.e. t, there is a subsequence i' (depending on t) for which

$$\int_{M_{i'}(t)} H^2 d\mathcal{H}^m \to 0.$$

Thus, passing to a further subsequence, we get varifold convergence to a stationary integral m-varifold $\tilde{M}$ (c.f. the Allard compactness theorem in Section REF). The unresolved issue with general (co)-dimensions is that all hell could break lose at the other times. For example, the M_i could look like two spheres very close together, joined by catenoidal necks, which move around very very fast.

However if $\tilde{M}$ is smooth and multiplicity 1, then "everything is nice."

5 renormalized MCF: analyzing singularities

Motivation: Consider a minimal variety $M^m \subset R^N$, with a singularity at $0 \in M$. The tangent cone at 0 is obtained by a sequence of dilations of M, which converge subsequentially to a minimal cone.

Alternatively, we can work in "conformal polar" coordinates given by

$$x \in R^N \mapsto (x/|x|, -\log|x|) \in S^{N-1} \times R,$$

so $M \mapsto \tilde{M}$. Now $\widetilde{\lambda M} = \tilde{M} - \log\lambda$, so the tangent cone in conformal polar coordinates is obtained via a sequence of translations, yielding a translation-invariant surface $\tilde{M}_\infty$.

Now consider $t \in R \mapsto M(t) \subset U$ an m-dimenensional MCF in R^N. Let $\mathcal{M} \subset\subset R^N \times I$ be the spacetime track. Without loss of generality suppose $(0,0) \in R^N \times I$ is a singularity. We consider spacetime dilations

$$\mathcal{D}_\lambda(x,t) := (\lambda x, \lambda^2 x),$$

constructed so that $\mathcal{D}_\lambda(\mathcal{M})$ is a MCF on $(\lambda U) \times (\lambda^2 I)$. The related conformal polar coordinates are

$$(x,t) \in R^N \times (-\infty, 0) \mapsto (\frac{x}{\sqrt{2|t|}}, -\frac{1}{2}\log|t|).$$

Definition 5.0.1. Given $(X,t) \in R^N \times (-\infty,0)$, the conformal polar coordinates $(\hat{X},\tau) \in R^N \times R$ are given by

$$(\hat{X},\tau) := \left(\frac{1}{\sqrt{2|t|}}X, -\frac{1}{2}\log|t|\right).$$

Conversely, we have

$$(X,t) = \left(\sqrt{2}e^{-\tau}\hat{X}, -e^{-2\tau}\right).$$

Lemma 5.1. *If* $\frac{\partial X}{\partial t} = H(X)$, *then*

$$\frac{\partial \hat{X}}{\partial \tau} = \hat{X} + H(\hat{X}).$$

Here $H(X), H(\hat{X})$ *are mean curvature vectors associated to* X, $\hat{X}$ *respectively.*

Proof. We have

$$\begin{aligned}\partial_\tau \hat{X} &= \partial_\tau t \partial_t \hat{X} \\ &= -2t\left(\frac{1}{2|t|}\frac{1}{\sqrt{2|t|}}X + \frac{1}{\sqrt{2|t|}}H(X)\right).\end{aligned}$$

□

Definition 5.1.1. An family of m-surfaces $M(t)$ in R^N which move with velocity $H + x^\perp$ is called the renormalized mean curvature flow (RMCF).

For example, a sphere of radius r centered at 0 moves like

$$\frac{dr}{dt} = -m/r + r.$$

Definition 5.1.2. A surface M is called a self-shrinker if $H = -x^\perp/2$. Equivalently, a family of surfaces $M(t)$ flowing by MCF are self-shrinkers if $H = \frac{-x^\perp}{2|t|}$, i.e. the associated RMCF $\hat{M}(\tau)$ is stationary.

Remark 5.2. RMCF satisfies a "super avoidance" principle, in the sense that distance increases exponentially.

Theorem 5.3 ("RMCF decreases weighted area")**.** *Let* $M(t) \subset R^N$ *be* m*-dimensional surfaces moving by RMCF. Define*

$$\phi = \frac{1}{(2\pi)^{m/2}} e^{-|x|^2/2}.$$

Then

$$\frac{d}{dt}\int_{M(t)} \phi = -\int_{M(t)} \phi |H + X^\perp|^2.$$

Remark 5.4. In Euclidean coordinates this becomes Huisken's monotonicity formula. Notice ϕ is normalized so that

$$\int_{R^m \times \{0\}} \phi = 1.$$

Proof. Let $\phi : R^N \to R$ be smooth. We have

$$\begin{aligned}\frac{d}{dt}\int_{M(t)} \phi d\mathcal{H}^m &= \int_{M(t)} (\nabla^\perp \phi - \phi \mathbf{H}) \cdot \partial_t \\ &= \int (\nabla^\perp \phi - \phi H) \cdot (H + X^\perp).\end{aligned}$$

Choose ϕ such that $\frac{\nabla \phi}{\phi} = -x$, with the correct normalization. □

Here is an alternate description of RMCF.

Theorem 5.5. *The m-dimensional RMCF is the same flow as $\partial_t = e^{\frac{-|x|^2}{2m}} \widetilde{H}$, where $\widetilde{H}$ is the mean curvature with respect to the metric $\tilde{g} = e^{\frac{-|x|^2}{2m}} \delta$.*

We use the RMCF to analyze singularities of the curve shortening flow. Let $\Gamma(t)$ be a MCF of embedded, closed curves, such that $\Gamma(0) \ni 0$, and let $\tau \in [0, \infty) \mapsto \hat{\Gamma}(\tau)$ be the associated RMCF. We observe that that

$$\hat{\Gamma}(\tau) \cap \partial B(0,1) \neq \emptyset$$

for all time. Otherwise, by the "super avoidance principle", the original Γ would avoid the spacetime origin $(0, 0)$.

In the same way that long-time existant MCF converges (up to choice of sequence) to geodesics, the RMCF will converge to geodesics in the weighted metric of Theorem weighted-metric-geodeics.

Theorem 5.6. *Let $\hat{\Gamma}_i(\tau) = \hat{\Gamma}(\tau - \tau_i)$, for some sequence $\tau_i \to \infty$. Then there is a subsequence i', such that for a.e. τ*

$$\hat{\Gamma}_{i'}(\tau) \to \textit{geodesic in metric } g = e^{-|x|^2/2}\delta$$

in $C^{1,\alpha}$. If the geodesic C is compact, then in Γ is "type-I", and $\hat{\Gamma}_i \to C$ convergences as C^∞ on compact subsets of spacetime.

Remark 5.7. In Euclidean coordinates the renormalized flows $\hat{\Gamma}_i(\tau)$ correspond to the dilated flows

$$\mathcal{D}_{e^{\tau_i}}(\Gamma).$$

In particular, convergence of $\hat{\Gamma}_{i'}$ to a geodesic in the renormalized metric, is precisely the statement that

$$\mathcal{D}_{e^{\tau_{i'}}}(\Gamma) \to \text{self-shrinker passing through spacetime origin.}$$

Remark 5.8. Many people classify singularities by "type-I" or "type-II". A singularity is type-I if the renormalized flow (centered at this singularity) has uniformly bounded curvature, and type-II otherwise. In the RMCF of a type-I singularity we get smooth convergence as $\tau \to \infty$, independent of any choice of sequence.

Notice that in original coordinates, a singularity at $(0, 0)$ is type-I iff

$$\limsup_{t \to 0_-} \sqrt{-t}|A| < \infty.$$

Proof. Let ϕ be as in Theorem REF, so that

$$\frac{d}{d\tau} \int_{\hat{\Gamma}(\tau)} \phi = - \int_{\hat{\Gamma}(\tau)} |H + x^\perp|^2 \phi.$$

As $\tau \to \infty$, we have that $\int_{\hat{\Gamma}(\tau)} \phi \to \theta$. Therefore, for any $a < b$,

$$\begin{aligned}
&\int_{\hat{\Gamma}(\tau_i+a)} \phi - \int_{\hat{\Gamma}(\tau_i+b)} \phi \to \theta - \theta = 0 \quad \text{as } i \to \infty \\
&= \int_{\tau_i+a}^{\tau_i+b} \int_{\hat{\Gamma}(\tau)} |H + x^{\perp}|^2 \phi ds dt \\
&= \int_a^b \int_{\hat{\Gamma}_i(\tau)} |H + x^{\perp}|^2 \phi ds dt.
\end{aligned}$$

Passing to a subsequence, we have

$$\sum_i \int_a^b \int_{\hat{\Gamma}_i(\tau)} |H + x^{\perp}|^2 \phi ds dt < \infty,$$

and in particular,

$$\int_{\hat{\Gamma}_i(\tau)} |H + x^{\perp}|^2 \phi ds \to 0 \quad \text{for a.e. } t.$$

So for a.e. τ, $\hat{\Gamma}_i(\tau)$ is locally bounded in $W^{2,2}(S^1)$. Pass to a further subsequence to obtain weak $W^{2,2}$ convergence $\hat{\Gamma}_i(\tau) \to C(\tau)$, where $C(\tau) \in W^{2,2}$. Hence we have $C^{1,\alpha}$ convergence to $C(\tau)$, and by weak convergence of the second derivatives

$$\int_{C(\tau)} |H + x^{\perp}|^2 \phi = 0.$$

Thus $C(\tau)$ is stationary for the RMCF.

If $C(\tau)$ is compact, then by smooth dependence on initial conditions $\hat{\Gamma}_i(\tau') \to C(\tau)$ smoothly for all $\tau' > \tau$. In particular, this implies the original MCF $\Gamma(\tau)$ is type-I. □

Remark 5.9. We should mention that smooth dependence on initial conditions can fail if C is non-compact. Here is an example. Let $C_i : [0, T] \to R^2$ be CFs, such that $C_i(0) \to L$ smoothly on compact subsets, where L is a line with multiplicity 2. We take each C_i to be the Grim-Reaper defined by

$$y = \frac{t}{\epsilon_i} - \frac{1}{\epsilon_i} \log(\cos \frac{x}{\epsilon_i}),$$

where $\epsilon_i \to 0$. Then for any $t > 0$, $C_i(t) \to \emptyset$. In fact in spacetime these converge to $L \times (-\infty, 0]$, but convergence is bad on $L \times \{0\}$.

These can arise as tangent flows of compact CSF, by considering the immersed figure-eight, where one loop is much smaller than the other. The smaller loop will pinch off before extinction as a Grim-Reaper (as a limit flow), or a double-density quasi-static line (as a tangent flow). Quasi-static means it exists as a static flow for $t < 0$, and disappears at $t = 0$.

To understand singularities of curve shortening flow we need to understand what the renormalized geodesic C can look like.

Theorem 5.10 (Albresch-Langer)**.** *Let C be a renormalized geodesic. If $0 \in C$, then C is a straight line. Any other C is contained in an annulus, and up to rotation is given by a 1-paramter family. If, further, C is embedded, then C must be a circle.*

Since we have C^1 convergence to C at t, embeddedness is preserved. So we can only end up with a circle or a line (possibly with multiplicity). If C is a circle then we have smooth convergence, and the CSF is very well behaved near the singularity.

If C is a line then we don't (a priori) have control over other times. To solve this we take weak limits.

6 level set approach to weak limits

We use Hausdorff convergence in spacetime. Recall that if K_i and K are closed subsets of R^N, then

$$\begin{aligned} &K_i \to K \text{ in the Hausdorff sense} \\ \iff &\text{a) each } x \in K \text{ is a limit of some } \{x_i \in K_i\}, \\ &\text{and b) given } \{x_i \in K_i\}, \text{ every subsequential limit lies in } K \\ \iff &\operatorname{dist}(\cdot, K_i) \to \operatorname{dist}(\cdot, K) \text{ uniformly on compact sets.} \end{aligned}$$

Further, any sequence K_i of closed subsets has a convergent subsequence in the Hausdorff sense. One can see this by, e.g. Arzela-Ascoli on $\operatorname{dist}(\cdot, K_i)$.

Theorem 6.1 ("limits don't fatten")**.** *Let Γ_i be a closed subsets of $R^2 \times [0, T]$ traced out by a CSF. Suppose*

$$\Gamma_i(0) \to \textit{line } L$$

in the Hausdorff sense. Then any subsequential limit of the Γ_i lies in $L \times [0, T]$.

Example 6.2. Suppose isntead of L we knew $\Gamma_i(0)$ converged to the curve S drawn below (IMAGE). Flow starting at S are not unique: we could instantaneously form a cusp, or the asymptotic lines could stay separate. Thus, depending on where we formed the cusp (or didn't), subsequential limits of Γ_i could approach a set of positive spacetime measure.

"Fattening" corresponds to non-uniqueness of the level set flow, in that this flow evolves to the region between all possible curve-shortening flows.

Proof. Let D be any disk in the complement $R^2 \sim L$. Then for large enough i, D is also in the complement of $\Gamma_i(0)$. Therefore by the avoidance principle, Γ_i avoids the parabaloid traced out in spacetime by moving ∂D according the CSF. But this holds for any disk D, and by making D sufficiently large, we can hit any spacetime point in $R^2 \times R \sim L \times R$. $\square$

We apply this to our previous result about RMCF. Let $\Gamma(t)$, and $\hat{\Gamma}(t)$ be as in the previous section. We know that for some subsequence $\hat{\Gamma}_{i'}$, and some time T, we have

$$\hat{\Gamma}_{i'}(T) \to L \text{ (with multiplicity m)}$$

in $C^{1,\alpha}$. Theorem REF says that for all $t > T$, we have $\hat{\Gamma}_i(t) \to L$ in the Hausdorff sense. I.e., "the line can't rotate."

In fact the multiplicity can't jump either (this isn't an issue if L was actually a circle). We know that for a.e. t, $\hat{\Gamma}_{i'}(t) \to L$ with some multiplicity $m(t)$. I claim that actually $m(t) = m$ for some constant independent of time and choice of subsequence.

Here's one way to see this. By C^1 convergence and monotonicity we have

$$m(t) = \lim_{i\to\infty} \int_{\hat{\Gamma}_i(t)} \phi = \lim_{t\to\infty} \int_{\hat{\Gamma}(t)} \phi.$$

Alternatively, one can use that $\#\hat{\Gamma}(t) \cap \partial B(0,1)$ is decreasing. We proved this for regular MCF, but the same result also holds for RMCF.

Theorem 6.3. *Let $t \in [0,T) \mapsto \Gamma_i(t)$ be a sequence of embedded (renormalized) curve shortening flows in $B(0,R_i)$, for $R_i \to \infty$. Suppose*

$$\Gamma_i(0) \to L = \textit{multiplicity-1 line.}$$

in C^1. Then $\Gamma_i \to L$ (with multiplicity 1) in C^1 on $B(0,R) \times [0,T)$, for every R.

Proof. Step 1: Let $a < \infty$. There exists an $I < \infty$ such that whenever $i > I$ and $t \in [0,T)$, then $\Gamma_i(t) \cap [-a,a]^2$ is a graph.

Proof of Step 1: Take $0 < \epsilon < a$. Then for i large

$$\Gamma_i(t) \cap [-a,a]^2 \subset \epsilon\text{-neighborhood of } L$$

by the Hausdorff convergence theoerem. Let S be the vertical line $\{x\} \times [-a,a]$. Then the number of intersections $\#(\Gamma_i(t) \cap S)$ is decreasing in time. We know $\#(\Gamma_i(t) \cap S) = 1$ at time 0 (for large i), so this number must be ≤ 1 for all time. But each $\Gamma_i(t)$ is an embedding of S^1, so in fact $\#(\Gamma_i(t) \cap S) \equiv 1$. END.

Step 2: The slopes $\to 0$ uniformly on $B(0,R) \times [0,T)$.

Proof of Step 2: Consider the tilted lines S_θ: PICTURE. As before, for $i > i(\theta)$ large, $\#(\Gamma_i(0) \cap S_\theta) = 1$, by uniform C^1 convergence to L. But $\#(\Gamma_i(t) \cap S_\theta)$ decreases in time, and so $\#(\Gamma_i(0) \cap S_\theta) \equiv 1$. □

Theorem 6.4. *Under the same hypothesis as the previous theorem, for any $\delta \in (0,T)$ and any R, the curvatures of $\Gamma_i \to$ uniformly on $B(0,R) \times [\delta,T)$.*

Proof. This is standard parabolic PDE. We know that $\Gamma_i(t) \cap [-a,a]^2$ is the graph of $u_i(\cdot,t) : [-a,a] \to [-a,a]$, where u_i satsifies the graphical (R)MCF equation EQREF, and $||u_i||_{C^1} \to 0$ as $i \to \infty$. It follows by parabolic Schauder that $||u_i||_{C^k} \to 0$ also. □

alternative proof. Suppose not. Then $\exists p_i \in \Gamma_i(t_i)$, for $\delta \leq t_i < T$, with p_i bounded and $\limsup_i k_i(p_i, t_i) > 0$. Wlog $p_i \to p$, $t_i \to t \in [\delta, T]$, and $k_i > 1/r > 0$.

Let C_i be the circle of radius r tangent to $\Gamma_i(t_i)$ at p_i, so $\#C_i \cap \Gamma_i(t_i) \geq 3$. PICTURE

Take $\tilde{C}_i$ to be the circle at time 0, such that under the (R)MCF $\tilde{C}_i \to C_i$ on $[0, t_i]$. Then $\#(\tilde{C}_i \cap \Gamma_i(0)) \geq 4$ by the strong maximum principle, since C_i touches $\Gamma_i(t_i)$ tangentially at p_i.

We can suppose that as $i \to \infty$, the $p_i \to p \in L$, $C_i \to C$, $\tilde{C}_i \to \tilde{C}$, and $\tilde{C} > C$ since $t_i \geq \delta > 0$. But since $\Gamma_i \to L$ in C^1, C lies tangent to L, and so $\tilde{C} \cap L$ is transverse (having cardinality 2). This is a contradiction. □

Now take $t \in [a, 0) \mapsto \Gamma(t)$ an embedded CSF in R^2. Let $\lambda_i \to \infty$, and consider the dilations

$$\Gamma_i(t) = \mathcal{D}_{\lambda_i}(\Gamma_i(t)), \quad \lambda_i^2 a \leq t < 0.$$

Assume for a.e. $t < 0$,

$$\Gamma_i(t) \to \text{multiplicity 1 line } L$$

in C^1. WLOG assume this holds for $t = -1$. So we have $\Gamma_i(-1) \to L$ in C^1, and therefore $k_i \to 0$ uniformly on $B(0, R) \times [-1, 0)$. This implies that the curvatures of the original Γ are uniformly bounded in a spacetime neighborhood of $(0, 0)$, and therefore $(0, 0)$ is a regular point.

In summary: if we take a $C^{1,\alpha}$ tangent flow as in blow-up theorem REF, then we can obtain either a circle (of multiplicity 1), a multiplicity 1 line, or a higher multiplicity plane. If we get a circle, then convergence is C^∞, and we've reached the conclusion of Grayson's theorem. If we have a multiplicity 1 line, then we are regular. We only need to rule out higher multiplicity to prove Grayson's theorem.

Lemma 6.5. *Let $t \in [0, T] \mapsto \Gamma_i(t)$ be a sequence of embedded CSFs in $B(0, R_i)$, for $R_i \to \infty$. Supppose*

$$\Gamma_i(t) \to \textit{mulplicitly } m \textit{ line } L$$

*in C^1, for $t = 0$ *and* $t = T$. In other words, $\Gamma_i(t)$ splits into m components, each of which $\to L$ in C^1.*

Then for $a < \infty$ and i large,

$$\Gamma_i(t) \cap [-a, a]^2 = \textit{union of } m \textit{ graphs}$$

for all $t \in [0, T]$.

The lemma shows we're well-controlled, with multiplicity, before and after, then we can't go to hell in between.

Proof. Let S be a vertical line. Since $\#(\Gamma(t) \cap S) = m$ at $t = 0, T$, and is non-increasing, then $\#(\Gamma(t) \cap S) \equiv m$. □

Theorem 6.6. *Let $\mathcal{M}$ be a MCF of hypersurfaces on $R^{n+1} \times [-T, 0)$, such every subsequential limit of $\mathcal{D}_\lambda \mathcal{M}$ (as $\lambda \to \infty$) converges to a multiplicity m plane. Suppose near $(0,0)$, $\mathcal{M}$ splits into m disjoint graphs. Then $m = 1$.*

For curve shortening flow we take the convergence of $\mathcal{D}_\lambda \mathcal{M}$ to be $C^{1,\alpha}$ as in the blow-up theorem. But for general flows we will show that convergence as Brakke flows suffices.

Proof. For simplicity we prove a contradiction for $m = 2$. The hypothesis implies each graphical component converges in C^∞ to 0 on compact subsets of $R^n \times (-\infty, 0)$. For $C^{1,\alpha}$ convergence in CSF this follows by the C^2 theorem and Schauder estimate. For Brakke convergence this follows from Brakke's regularity.

Let $p_j(t)$ $(j = 1, 2)$ be the point on each graph closest to 0, and let $\delta(t)$ be the parabolic-scale-invariant quantity

$$\delta(t) = \frac{|p_1(t) - p_2(t)|}{\sqrt{|t|}}.$$

Since in the RMCF we converge to a plane as $\tau \to \infty$, we muat have $\delta(t) \to 0$ as $t \to 0$.

Choose t_i so that $\delta(t_i) = \max_{t \geq t_i} \delta(t)$, and let $\lambda_i = \frac{1}{\sqrt{|t_i|}}$. After passing to a subsequence, we have convergence of $\mathcal{D}_{\lambda_i} \mathcal{M}$ to a multiplicity m plane.

Let $u_i > v_i$ be the graphs of $\mathcal{D}_{\lambda_i} \mathcal{M}$ on domains Ω_i exhausing $R^n \times (-\infty, 0)$. Each u_i and v_i satisfy the MCF equation

$$u_t = \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u,$$

and so their difference satisfies a linear PDE

$$(u - v)_t = \Delta(u - v) + a_{ij} D_{ij}(u - v) + b_i D_i(u - v) + c(u - v)$$

where the coefficients a_{ij}, b_i and c tend to 0 in C^∞, as $i \to \infty$.

Clearly $u_i(0,t) - v_i(0,t) \sim |p_1(t) - p_2(t)|$. Define

$$w_i = \frac{u_i - v_i}{u_i(0,-1) - v_i(0,-1)}.$$

By our choice of t_i, we have

$$\frac{w_i(0,t)}{\sqrt{|t|}} \leq (1 + \epsilon_i) w_i(0,-1) = (1 + \epsilon_i),$$

for $t \in [-1, 0)$, and with $\epsilon_i \to 0$.

Recall that w_i are decreasing. The Harnack inequality ("function at earlier time bounded by later times") implies w_i is uniformly bounded above on compact subsets of spacetime, and so by Schauder we have uniform bounds on all derivatives. After passing to a subsequence, we obtain that $w_i \to w$ in C^∞, where w is defined on all of $R \times (-\infty, 0)$ and satisfies the usual heat equation $w_t = \Delta w$.

By inequality REF we also have

$$0 \leq \frac{w(0,t)}{\sqrt{|t|}} \leq w(0,-1) = 1$$

for all $t \in [-1, 0)$. But this violates the strong maximum principle. To see this, use a comparison function like

$$\epsilon e^{-t/\ell^2} \cos(x/\ell)$$

and observe that $\sqrt{|t|} < \epsilon e^{-t/\ell^2}$ for sufficiently small t. □

7 varifolds

Definition 7.0.1. An m-varifold in $U \subset R^N$ is a Random measure on $U \times Gr(m, N)$. To each varifold V there is an associated Radom measure μ_V on U given by $\mu_V = \pi_{\#} V$, where $\pi : U \times Gr(m, N) \to U$ is the projection.

The total mass of V in U is defined to be $||V||_U := \mu_V(U)$.

Lemma 7.1. *If M, M' are C^1 m-manifolds in U, and*

$$Z = \{x \in M \cap M' : Tan(M, x) \neq Tan(M', x)\},$$

then $\mathcal{H}^m Z = 0$.

Proof. This follows directly from the implicit function Theorem. □

Corollary 7.2. *Suppose $S \subset \cup_i M_i$, and also $S \subset \cup_i M'_i$, with M'_i, M_i being C^1 m-manifolds in U. Let T, T' be defined on S by*

$$T(x) = Tan(M_i, x), \quad (x \in M_i \sim \cup_{j \neq i} M_j)$$
$$T'(x) = Tan(M'_i, x), \quad (x \in M'_i \sim \cup_{j \neq i} M'_j).$$

Then $T = T'$ $\mathcal{H}^m$-a.e. $x \in S$.

Definition 7.2.1. Let S be a Borel subset of an m-dimensional C^1 manifold $M \subset U$. Then the varifold V_S associated to S is defined by

$$\int f dV_S := \int_S f(x, Tan(M, x)) d\mathcal{H}^m$$

for any $f \in C^0_c(U \times Gr(m, N))$. Alternatively, if $A \subset U \times Gr(m, N)$, then

$$V_S(A) := \mathcal{H}^m(\pi\{(x, T) \in A : x \in S, T = Tan(M, x)\}).$$

Definition 7.2.2. An integral m-varifold V in U is a a countable sum

$$V = \sum_i V_{S_i}$$

where each S_i is a Borel subset of some C^1 m-manifold $M_i \subset U$, and V_{S_i} the associated varifold as per definition REF.

ALTERNATIVELY, we can define an integral m-varifold as follows: Let $\theta \in L^1_{loc}(U, Z_+; \mathcal{H}^m)$ be such that

$$S = \{\theta > 0\} \subset Z \cup (\cup_i M_i)$$

where $\mathcal{H}^m(Z) = 0$ and each M_i is a C^1 m-manifold in U. Then the integral varifold V_θ is defined by

$$\begin{aligned}\int f dV\theta &:= \int \theta(x) f(x, T(x)) d\mathcal{H}^m \\ &= \sum_i \int_{M_i \sim \cup_{j<i} M_j} \theta(x) f(x, Tan(M_i, x)) d\mathcal{H}^m.\end{aligned}$$

where T as in Corollary REF.

The two definitions are related by

$$\theta = \sum_i 1_{S_i}.$$

If V is an integral varifold, with $\mu_V(U) < \infty$, then for every $\epsilon > 0$ there are finitely many disjoint, compact C^1 m-manifolds M_i with boundary such that

$$||V - V_{\sum_i k_i M_i}||_U < \epsilon.$$

Definition 7.2.3. Let X be a compactly supported C^1 vector field, and ϕ_t the corresponding integral flow. For general varifold V, define the first-variation $\delta V : \mathfrak{X}(U) \to R$ by

$$\begin{aligned}\delta V(X) &:= \frac{d}{dt}|_{t=0} ||\phi_{t\#} V||_U \\ &= \int_{x \in U, T \in Gr(m,N)} div_T(X) dV(x, T)\end{aligned}$$

where we define $div_T(X) = (\sum_i e_i \cdot \nabla_{e_i} X)$, with e_i being any ON basis of T. The second equality follows directly from the usual first-variation calculations.

If $V = V_\theta$ is integral we have the alternative description

$$\delta V(X) = \int_{x \in U} div_{Tan(V,x)} X d\mu_V(x).$$

Remark 7.3. Notice that $div_T(X)$ is continuous in (x, T), so trivially if $V_i \rightharpoonup V$, then $\delta V_i \rightharpoonup \delta V$ also.

Remark 7.4. "varifold boundary is like boundary interpreted in C^1-sense, whereas current boundary is boundary in C^0 sense. In particular, the polygons have 0 boundary as currents, but non-zero boundaries as varifolds.

We say that V has locally bounded first-variation if δV is locally bounded, in the sense that

$$|\delta V(X)| \leq C_K |X|_{C^0}, \quad \text{spt} X \subset K \subset\subset U.$$

In this case the Reisz representation theorem gives a Radon measure λ on U, and a λ-measurable unit vector field Λ such that

$$\delta V(X) = \int_U X \cdot \Lambda d\lambda.$$

Write $\lambda = \lambda_{ac} + \lambda_{sing}$ for $\lambda_{ac} << \mu_V$ and $\lambda_{sing} \perp \mu_V$. Then

$$\begin{aligned}\delta V(X) &= \int X \cdot \Lambda \frac{d\lambda_{ac}}{d\mu_V} d\mu_V + \int X \cdot \Lambda d\lambda_{sing} \\ &=: -\int X \cdot H d\mu_V + \int X \cdot \nu d\sigma.\end{aligned}$$

where we've defined $H := -\Lambda \frac{d\lambda_{ac}}{d\mu_V}$ to be the generalized mean curvature, and $\sigma := d\lambda_{sing}$ the measure-theoretic boundary. Note that this holds for any general varifold with locally bounded first variation.

The following compactness theorem is the analogue of the Federer-Fleming compactness for integral currents. Actually the theorem is more of a closure theorem, as the space of Radon measures is already weakly compact. Allard's theorem states that the space of integral m-varifolds is closed, in the space of Radon measures on $U \times Gr(m, N)$.

Theorem 7.5 (Allard's theorem)**.** *If V_i are integral varifolds such that*

$$\sup_i |V_i|(K) + |\delta V_i|(K) \leq C_K, \quad \forall K \subset\subset U$$

then there is an integral varifold V such that $V_i \to V$ as varifolds (i.e. as Radon measures on $U \times Gr(m, N)$), and $|V|_K + |\delta V|_K \leq C_K$.

Notice that the condition $|\delta V_i(X)| \leq C_K |X|_0$ is equivalent to

$$\int_K |H_i| d\mu_V + \int_K d\sigma_i \leq C_K.$$

Remark 7.6. In general we have no control of the (dis)appearance of generalized boundaries. For example, consider a sequence of polygons approaching a circle. For each polygon, $H = 0$ but $\sigma \neq 0$, while for the circle $H \neq 0$ but $\sigma = 0$.

Conversely, one could take a sequence of squashed circles approaching a multiplicity-2 line-segment. Each circle has $H \neq 0$, $\sigma = 0$, but the multiplicity-2 line segment will have $H = 0$, $\sigma \neq 0$.

We have the refinement

Theorem 7.7. *Let V_i be integral m-varifolds with locally bounded first variation, and no generalized boundary, i.e.*

$$\delta V_i(X) = -\int X \cdot H_i d\mu_{V_i}.$$

Suppose that for every compact $K \subset\subset U$ we have

$$\sup_i |V_i|(K) \leq C_K < \infty,$$

and

$$\int_K |H_i|^2 \leq C_K < \infty, \quad \text{spt}X \subset K \subset\subset U.$$

(actually we could take the L^p norm for any $p > 1$)

Then there is an integral m-varifold V, such that $V_i \to V$ as varifold. V has locally bounded first variation and no generalized boundary, i.e. $\delta V(X) = -\int H \cdot X d\mu_V$, and

$$\int H_i \cdot X d\mu_{V_i} \to H \cdot X d\mu_V.$$

In fact, if X is continuous, compactly supported in $U \times Gr(m, N)$, then

$$\int H_i(x) \cdot X(x, Tan_{V_i}(x)) d\mu_{V_i} \to \int H(x) \cdot X(x, Tan_V(x)) d\mu_V.$$

Proof. For each V_i and any X supported in K we have

$$\begin{aligned} |\delta V_i(X)| &\leq \left(\int |H_i|^2 d\mu_{V_i}\right)^{1/2} \left(\int |X|^2 d\mu_{V_i}\right)^{1/2} \\ &\leq C_K^{1/2} ||X||_{L^2(\mu_{V_i})}. \end{aligned}$$

Therefore by weak convergence the same bound holds for V:

$$|\delta V(X)| \leq C_K^{1/2} \left(\int |X|^2 d\mu_V\right)^{1/2}.$$

So the Reisz representation Theorem implies there's a vector field $H \in L^2_{loc}(U, R^N; \mu_V)$ so that

$$\delta V(X) = -\int H \cdot X d\mu_V.$$

□

8 Brakke flows

Definition 8.0.1. An m-dimensional Brakke flow in $U \subset R^N$ is a 1-parameter family of measures $t \in I \mapsto \mu(t)$, such that:

a) for a.e. t, $\mu(t) = \mu_{V(t)}$ for some integral varifold $V(t)$ (i.e. $\mu(t) = \pi_\# V(t)$, where $\pi : R^N \times Gr(m, N) \to R^N$ is the natural projection), with $H_V \in L^2_{loc}(U)$, and no boundary term in the first variation.

b) for any $f \in C^2_c(U \times [c, d])$, $f \geq 0$, and $[a, b] \subset [c, d]$, $\mu(t)$ satisfies

$$\int f(\cdot, b) d\mu(b) - \int f(\cdot, a) d\mu(a) \leq \int_a^b \int (-H^2 f + H \cdot \nabla^\perp f + \partial_t f) d\mu(t) dt. \quad (1)$$

Actually you could just use ∇f, as Brakke (REF) later proved (after making this definition) that $H \perp T_x M$ μ_V-a.e. Also I think in this definition you don't need your test function to be C^2, probably just C^1. One can alternately define it in terms of difference quotients (see ILMANEN), in which case C^2 is required.

Remark 8.1. If M_t is a smooth MCF, then equation REF holds with equality. The inequality is required for compactness. For example, by a sequence of scaled Grim-Reapers would converge to the flow which is a multiplicity-2 line for $t < 0$, and the empty set for $t > 0$.

An unfortunate side-effect of this inequality is that the definition allows for instantaneous vanishing, i.e. at some t_0 we could simply define $\mu(t) = 0$ for $t > t_0$. This is an intrinsic non-uniqueness built into the definition. However we can ignore this issue by restricting to various subclasses of Brakke flows. E.g. those obtained by elliptic regularization.

Proposition 8.2 (mass bounds)**.** *Let $\mu(t)$ be an m-dim Brakke flow. Define*

$$\phi(x, t) = R^2 - |x|^2 - 2mt.$$

then $\int \phi_+^4 d\mu(t)$ is decreasing in time.

Proof. We have $\nabla \phi_+^4 = 4\phi_+^3(-2x)$, and

$$div_V(\nabla \phi_+^4) = 24\phi_+^2 |x^T|^2 - 8n\phi_+^3.$$

For a.e. t we have $\mu(t) = \mu_{V(t)}$. Since $V(t)$ has no generalized boundary, we have

$$-\int 8n\phi_+^3 \mu_{V(t)} \leq \int div_{V(t)}(\nabla \phi) = -\int H_{V(t)} \cdot \nabla \eta d\mu_{V(t)}.$$

And therefore

$$\begin{aligned} \int \phi_+^4 d\mu(b) - \int \phi_+^4 d\mu(a) &\leq \int_a^b \int -H^2 \phi_+^4 + H \cdot \nabla \phi_+^4 - 8n\phi_+^3 d\mu_{V(t)} dt \\ &\leq \int_a^b \int 8n\phi_+^3 - 8n\phi_+^3 d\mu_{V(t)} dt. \end{aligned}$$

□

Corollary 8.3. *If $[0,T) \ni t \mapsto \mu(t)$ is a Brakke flow in $U \subset R^N$, with $T < \infty$, then*

$$\sup_{t\in[0,T)} \mu(t)K < \infty, \quad \forall K \subset\subset U.$$

Consequently, $\mu(T) = \lim_{t\to T_-} \mu(t)$ exists.

Proof. The second part follows by the next theorem. □

Definition 8.3.1. We say a non-negative function $\phi \in C_c^2(U)$ is "nice" if $\{\phi = 0\} \subset \{D\phi = 0\}$. For example, given a "not-nice" ϕ, we could take ϕ^r for any $r > 1$.

Theorem 8.4. *Let μ_t be a Brakke flow on $U \times [a,b]$. For any nice $\phi \in C_c^2(U)$, there is a constant C_ϕ such that*

$$t \mapsto \mu(t)\phi - C_\phi t$$

is decreasing in time. C_ϕ depends on ϕ, and $\sup_{t\in[a,b]} \mu(t)\mathrm{spt}\phi$.

As a Corollary, we have that for any $t \in [a,b]$,

$$\lim_{\tau\to t_-} \mu(\tau) \geq \mu(t) \geq \lim_{\tau\to t_+} \mu(\tau).$$

Proof. Let $\phi \in C_c^2(U)$, and take $[c,d] \subset [a,b]$. Then by definition of Brakke flow we have, using Proposition mass-bounds,

$$\begin{aligned}\int \phi d\mu(d) - \int \phi d\mu(c) &\leq \int_c^d \int -H^2\phi + H \cdot \nabla^\perp \phi d\mu(t)dt \\ &\leq \iint -H^2\phi + \frac{1}{2}H^2\phi + \frac{1}{2}\frac{|\nabla\phi|^2}{\phi} d\mu(t)dt \\ &\leq (d-c)C|\phi|_{C^2(U)}\end{aligned}$$

The second part of the the Theorem follows by immediately from the first, using a straightforward approximation argument. □

Theorem 8.5 (H bounds)**.** *We have, for any compact $K \subset\subset U$,*

$$\int_a^b \int_K H^2 d\mu(t)dt < C < \infty,$$

where the constant depends only on $\sup_{t\in[a,b]} \mu(t)(K)$ and $|b-a|$. Of course these are finite by Propposition mass-bounds.

Proof. Suppose $\phi \in C_c^2(U)$ is nice. We have

$$\frac{|\nabla\phi|^2}{\phi} \in C_c^1(U).$$

We calculate

$$\int \phi d\mu(a) - \int \phi d\mu(b) \geq \int_a^b \int (\phi H^2 - H \cdot \nabla^\perp \phi) d\mu(t) dt$$
$$\geq \frac{1}{2} \iint \phi H^2 - \frac{1}{2} \frac{|\nabla \phi|^2}{\phi} d\mu(t) dt.$$

And therefore, using the mass bounds of Proposition REF,

$$\int_a^b \int H^2 \phi d\mu(t) dt \leq \int \phi d\mu(a) - \int \phi d\mu(b) + \frac{1}{2} \iint \frac{|\nabla \phi|^2}{\phi} d\mu(t) dt$$
$$\leq C_{\mathrm{spt}\phi} \left(|\phi|_{C^0} + \left| \frac{|\nabla \phi|^2}{\phi} \right|_{C^0} (b-a) \right)$$
$$\leq C_{\mathrm{spt}\phi} ||\phi||_{C^2} (1 + (b-a)).$$

□

Theorem 8.6 (compactness of integral Brakke flows)**.** *Suppose $t \in [a,b] \mapsto \mu_i(t)$ is a sequence of integral Brakke flows, such that*

$$\sup_i \sup_{t \in [a,b]} \mu_i(t)(K) \leq C_K < \infty, \quad \forall K \subset\subset U.$$

Then there is an integral Brakke flow $\mu(t)$ on $U \times [a,b]$, and a subsequence i', so that $\mu_{i'}(t) \to \mu(t)$ as Radon measures on U for all $t \in [a,b]$. Further, for a.e. $t \in [a,b]$, after passing to a further subsequence i'' (depending on t), we have

$$V_{i''}(t) \to V(t)$$

as varifolds. Here $V_{i''}(t)$ and $V(t)$ are the integral m-varifolds associated to $\mu_{i''}(t)$ and $\mu(t)$ respectively.

Proof. Let $\phi \in C^2_c(U)$, $\phi \geq 0$. Recall that

$$L_i^\phi(t) := \int \phi d\mu_i(t) - C_\phi C_{\mathrm{spt}\phi} t$$

is decreasing in t, and uniformly bounded in i. (The constants depend only on ϕ). Pass to a subsequence (depending on ϕ) to obtain

$$L_i^\phi(t) \to L(t)$$

for some decreasing function $L(t)$. In particular, $\int \phi d\mu_i(t)$ converges to some limit for every t.

By diagonalizing we can therefore assume that, for every ϕ in a countable dense subset of $C^2_c(U)$, $\int \phi d\mu_i(t)$ converges for all t. Therefore we can assume $\mu_i(t) \rightharpoonup \mu(t)$ for all t.

Let

$$M^\phi(t) = \liminf_i \int \phi H_i^2 - \nabla^\perp \phi \cdot H_i d\mu_i(t).$$

By Fatou we have

$$\int \phi d\mu_i(a) - \int_a^b \phi d\mu_i(b) \geq \int_a^b M^\phi(t) - \int_a^b \int \partial_t \phi d\mu dt.$$

And so by the uniform mass bounded for a.e. t we must have $M^\phi(t) < \infty$.

Let $K \subset\subset U$, and then for all i,

$$\sup_{t\in[a,b]} \mu_i(t)(K) < C < \infty$$
$$\int_a^b \int_K H_i^2 d\mu_i(t) dt \leq D < \infty.$$

By Propositions massbounds, H-bounds.

Let $f_K(t) = \liminf_i \int_K H_i^2 d\mu_i(t)$. Then by Fatou's Lemma

$$\int_a^b f(t)dt \leq \liminf_i \int_a^b \int_K H_i^2 d\mu_i(t) dt \leq D.$$

In particular, for a.e. t, $f(t) < \infty$.

Repeating the above argument for any exhaustion $K_1 \subset K_2 \subset \cdots U$, we have: for a.e. $t \in [a,b]$, $f_K(t) < t$ for every $K \subset\subset U$.

We now fix a t such that: a) for every i, $\mu_i(t) = \mu_{V_i(t)}$ where $V_i(t)$ is an integral m-varifold, b) $f_K(t) < \infty$ for all $K \subset\subset U$, c) $M^\phi(t) < \infty$ for all ϕ is a countably dense subset $\mathcal{C}$ of $C_c^2(U)$.

We can choose a subsequence i' (depending on t) so that:

$$\sup_{i'} \int_K \mu_{V_{i'}(t)} + \int_K H_{i'}^2 \mu_{V_{i'}}(t) < \infty, \quad \forall K \subset\subset U.$$

and for every $\phi \in \mathcal{C}$,

$$M^\phi(t) = \lim_{i'} \int \phi H_{i'}^2 - \nabla^\perp \phi \cdot H_{i'} d\mu_{i'}(t).$$

By the L^p-compactness of integral varifolds, there is an integral m-varifold $V(t)$, and a further subsequence, so that $V_{i'}(t) \rightharpoonup V(t)$, and

$$\int H_{V_{i'}(t)} \cdot X d\mu_{V_{i'}(t)} \to \int H_{V(t)} \cdot X d\mu_{V(t)}, \quad \forall X \in C_c(U).$$

And for this t, and every $\phi \in \mathcal{C}$,

$$\int \phi H_{V(t)}^2 - \nabla^\perp \phi \cdot H_{V(t)} d\mu_{V(t)} \leq M^\phi(t).$$

But we note that $V(t)$ depends only on the sequence i, as any integral varifold depends only on $\mu_{V(t)}$, and $\mu_{V_i} \rightharpoonup \mu_V$. So the above equation EQREF is independent of i', and therefore holds for almost every t, and every $\phi \in \mathcal{C}$.

We deduce that, for $\phi \in \mathcal{C}$, and

$$\int \phi d\mu(a) - \int \phi d\mu(b) \geq \int_a^b \int \phi H^2 - \nabla^\perp \phi \cdot H - \partial_t \phi d\mu(t) dt.$$

By approximation in fact this holds fore very $\phi \in C^2_c(U)$, and the proof is complete. □

9 elliptic regularization

One should keep in mind that, like integral currents, a Brakke flow is merely a framework, and says nothing by itself about existence. Any smooth MCF is trivially a Brakke flow, but the question remains how to extend the weak flow past singularities. Ilmanen created a very powerful existence Theorem called elliptic regularization, which produces Brakke flows satisfying several nice additional properties (e.g. no sudden vanishing).

A particularly nice feature these flows have is that they are limits of smooth flows, interpreted in a suitable sense. This allows for many proofs to be greatly simplified.

Theorem 9.1. *Let $z : R^N \times R \to R$ be the height function (i.e. so $z(x,h) := h$), and $e = \nabla z$ the upward unit normal. Let M be an $(n+1)$-surface in $R^N \times R$, and $\lambda > 0$. Then M is stationary for the functional*

$$\mathcal{I}_\lambda : M \mapsto \int_M e^{-\lambda z} d\mathcal{H}^{n+1}$$

if and only if $t \mapsto M - \lambda t e$ is a mean curvature flow in $N \times R$.

Proof. Let $s \mapsto M_s$ be a deformation, wtih $M_0 = M$ and velocity X. Then

$$\begin{aligned} \frac{d}{ds}|_{s=0} \mathcal{I}_\lambda[M_s] &= \int_M e^{-\lambda z} div_M(X) + \nabla(e^{-\lambda z}) \cdot X \\ &= \int_M e^{-\lambda z}(div_M(X^\perp) - \lambda e^\perp \cdot X) \\ &= \int_M e^{-\lambda z} X \cdot (-H - \lambda e^\perp). \end{aligned}$$

So M is stationary iff $H = -\lambda e^\perp$ iff

$$\partial_t^\perp = -\lambda e = H.$$ □

Here is the strategy. Let Σ^m be a compact surface in R^N. Let $M_\lambda^{m+1} \subset R^N \times R$ minimize $\mathcal{I}_\lambda$, subject to $\partial M_\lambda = \Sigma \times \{0\}$. One can easily show that M_λ lives in $R^N \times R_+$, since the projection

$$R^N \times R_- \to R^N \times \{0\} : (x, h) \mapsto h$$

is distance decreasing.

So $\mu_\lambda(t) := M_\lambda - \lambda e t$ is a mean curvature flow. We will take $\lambda \to \infty$, and obtain a subsequence (as Brakke flows) of the $\{\mu_\lambda(t)\}_\lambda$ converging to a limit flow $\mu(t)$. The flow $\mu(t)$ will have the properties:

- $\mu(t) = \Sigma(t) \times R$, for a.e. t, and hence $t \mapsto \Sigma(t)$ will be an m-dim MCF in R^N.
- $\Sigma(0) = \Sigma$.

Towards this goal we need local mass bounds on each M_λ. Let $M_\lambda(a,b) = M_\lambda \cap \{a < z < b\}$, and $S_\lambda(a) = M_\lambda \cap \{z = a\}$.

Lemma 9.2. *Let M_λ be stationary for $\mathcal{I}_\lambda$. Then*

$$|M_\lambda(a,b)| \leq (1/\lambda + (b-a))|\Sigma|.$$

In particular, by Proposition mass-bounds the Brakke flow $M_\lambda(t)$ has uniform mass bounds in spacetime, independent of λ.

Proof. Let ν be the cornomal of $S_\lambda(z)$, which points in the $\nabla z = e$ direction. We have

$$\begin{aligned} 0 &= \int_{M_\lambda(a,b)} div_M(e) \\ &= \lambda \int_{M_\lambda(a,b)} |e^\perp|^2 + \int_{S_\lambda(b)} |e^T| - \int_{S_\lambda(a)} |e^T|. \end{aligned}$$

and therefore

$$z \mapsto \int_{S_\lambda(z)} |e^T|$$

is decreasing in z, with equality over $[a,b]$ iff $M_\lambda(a,b)$ is translationally invariant.

We calculate

$$\begin{aligned} |M_\lambda(a,b)| &= \int_{M_\lambda(a,b)} |e^\perp|^2 + |e^T|^2 \\ &\leq \frac{1}{\lambda} \int_{S_\lambda(0)} |e^T| + \int_{M_\lambda(a,b)} |e^T|^2 \\ &\leq \frac{1}{\lambda}|\Sigma| + \int_a^b \int_{S_\lambda(z)} |e^T| d\mathcal{H}^{m-1} dz \\ &\leq (\frac{1}{\lambda} + (b-a))|\Sigma|. \end{aligned}$$

□

We can thus obtain a subsequential limit $\mu_{\lambda_i} \to \mu$ as Brakke flows, where $\lambda_i \to \infty$.

Lemma 9.3. *$\mu(t)$ is translationally-invariant, in the sense that for any non-negative $\phi \in C^2_c(R^N \times R)$, and a.e. t, we have*

$$\mu(t)\phi = \mu(t)\phi(\cdot, \cdot - h).$$

Proof. Given ϕ, set $\phi^h(x,z) := \phi(x, z-h)$. Since $\mu_\lambda(t)$ moves by translation, we have

$$\mu_\lambda(t)\phi^h = \mu(t + h/\lambda)\phi.$$

Let ϕ be nice, compactly supported, non-negative in R^{N+1}. There is a constant C independent of λ so that $t \mapsto \mu_\lambda(t)\phi - Ct$ is decreasing in t. So if $t < t + h/\lambda < s$, then

$$\begin{aligned}
\mu_\lambda(t)\phi - Ct &\geq \mu_\lambda(t + h/\lambda)\phi - C(t + h/\lambda) \\
&= \mu_\lambda(t)\phi^h - C(t + h/\lambda) \\
&\geq \mu_\lambda(s)\phi - Cs.
\end{aligned}$$

Let $\lambda \to \infty$, and since $\mu_\lambda(t) \rightharpoonup \mu(t)$ for *all* times, we have

$$\mu(t)\phi \geq \mu(t)\phi^h \geq \mu(s)\phi - C(s-t)$$

for every $t < s$. Take $s \to t$, and we have $\mu(t)\phi \geq \mu(t)\phi^h \geq \mu(t^+)\phi$. But $\mu(t^-) = \mu(t) = \mu(t^+)$ outside a countable set. □

Lemma 9.4. *We have that* $\mu(0) = \mu_{\Sigma \times R^+}$.

Proof. Let $\pi_{R^N \times \{b\}}$ be orthogonal projection onto $R^N \times \{b\}$. PICTURE

We calculate that

$$\begin{aligned}
|\pi_{R^N \times \{b\}} M_\lambda(0,b)| &= \int_{M_\lambda(0,b)} |e^\perp| \\
&\leq |M_\lambda(0,b)|^{1/2} \left(\int_{M_\lambda(0,b)} |e^\perp|^2 \right)^{1/2} \\
&\leq C|\Sigma|^{1/2} \left(\frac{|\Sigma|}{\lambda} \right)^{1/2} \\
&\to 0 \quad \text{as } \lambda \to \infty,
\end{aligned}$$

having used (the calculations of) mass-bounds lemma. Similarly, if $A_\lambda(0,b)$ is the region bounded by $M_\lambda(0,b)$ and $\pi_{R^N \times \{b\}} M_\lambda(0,b)$, then

$$|A_\lambda(0,b)| \leq (b-a)|\pi M_\lambda(0,b)| \to 0.$$

Therefore $M_\lambda(0,b) \to \Sigma \times [0,b]$ in the flat norm. By lower semi-continuity of mass, we obtain

$$\begin{aligned}
|\Sigma \times [0,b]| &\leq \liminf_{\lambda \to \infty} |M_\lambda(0,b)| \\
&\leq \limsup_{\lambda \to \infty} (1/\lambda + b)|\Sigma| \\
&= b|\Sigma| \\
&= |\Sigma \times [0,b]|.
\end{aligned}$$

So in fact the masses converge also. This implies the associated Radon measures converge: $\mu_{M_\lambda(0,b)} \rightharpoonup \mu_{\Sigma \times [0,b]}$. But $\mu_\lambda(0) \equiv \mu_{M_\lambda}$. We deduce that any (subsequential) limit of $\mu_\lambda(0)$ must be $\mu_{\Sigma \times R^+}$. □

The above two lemmas establish the required properties of the limit flow $\mu(t)$. We show that $t \mapsto \Sigma(t)$ is non-vanishing.

Definition 9.4.1. The support of $t \mapsto \mu(t)$ in U is the smallest (relatively) closed subset of $U \times [0, \infty)$ containing $\cup_{t\in[0,\infty)}\mathrm{spt}(\mu(t)) \times \{t\}$.

The following fact follows by the monoconicity formula, introduced in the next section; it is a direct consequence of upper-semicontinuity of Gaussian density.

Fact 9.5. *If Brakke flows converge, then the supports converge as sets. I.e.* $\lim \mathrm{spt} = \mathrm{spt} \lim$.

Proposition 9.6. *The Brakke flow $t \mapsto \Sigma(t)$ does not instantly vanish.*

Proof. Consider a line segment γ transversely intersecting Σ once, with $\{p, q\} = \partial\gamma$. Choose little circles S_p and S_q centered at p, q respectively, and disjoint from Σ. Write $P_{p,\lambda}$ and $P_{q,\lambda}$ for the minimizers of $\mathcal{I}_\lambda$ with boundaries S_p, S_q. We know these are both disjoint from M_λ.

As a boundaryless surface in $\{z > 0\}$, the translator $M_\lambda - \lambda et$ must meet γ until it hits either p or q. Therefore by disjointness it must meet γ until either $P_{p,\lambda} - \lambda et$ hits p, or $P_{q,\lambda} - \lambda et$ hits q.

We deduce that, for some $\epsilon > 0$, $(M_\lambda - et) \cap \gamma \neq \emptyset$ for $t \in [0, \epsilon)$, and for all λ. Therefore in the limit $\Sigma(t) \times R$ must meet γ for all $t \in [0, \epsilon)$ also, by set-convergence of supports. This shows $\Sigma(t)$ cannot instantly vanish. $\square$

Remark 9.7. By taking the limit of M_λ's as integral currents in spacetime, we can endow our resulting Brakke flow with a current structure.

Another way of showing non-vanishing is to take the integral current limit of the M_λ, and obtain an integral current supported in the spt of the Brakke flow limit, with $\partial = \Sigma = \{0\}$.

10 monotonicity and entropy

Given $M^m \subset R^N$, then the (m-dimensional) Gaussian density is defined to be

$$F_m(M) = \frac{1}{(4\pi)^{m/2}} \int_M e^{-|x|^2/4} d\mathcal{H}^m.$$

Similarly we can define $F_m(\mu)$ for a Radon measure μ in the obvious way. The Gaussian density is the crucial ingredient behind (R)MCF monotonicity.

Recall that if M_t flows by RMCF, so $vel = X + H(X)$, then $F(M_t)$ is decreasing in time. Precisely, whenever $a \leq b \leq 0$,

$$F(M_b) - F(M_a) \leq -\int_a^b \int_{M_t} \frac{1}{(4\pi)^{m/2}} e^{-|x|^2/4} |H + (x/2)^\perp|^2 d\mu(x) dt.$$

In fact for a smooth RMCF this is an equality, but one can formulate weak RMCF (by, e.g., applying conformal polar coords to a Brakke flow), and for a weak RMCF there is only inequality.

By transforming this back to Euclidean coordinates, or by direct calculation, one can show the following: define

$$\rho(x,t) = \frac{1}{(4\pi|t|)^{m/2}} e^{\frac{-|x|^2}{4|t|}}.$$

Theorem 10.1 (Gaussian monotonicity)**.** *If $\mu(t)$ is an m-Brakke flow in R^N, then for $a \leq b \leq 0$, we have*

$$\int \rho d\mu(b) - \int \rho d\mu(a) \leq -\int_a^b \int \rho \left| H + \frac{x^\perp}{2|t|} \right|^2 d\mu(t)dt.$$

If $\mu(t)$ is a smooth flow then we have equality.

Definition 10.1.1. If $\mathcal{M}$ is a Brakke flow, and $X = (x_0, t_0)$, then the Gaussian density ratio is defined to be

$$\Theta(\mathcal{M}, X, r) = \int \frac{1}{(4\pi r^2)^{m/2}} e^{\frac{-|x-x_0|^2}{4r^2}} d\mu(t_0 - r^2)(x).$$

(We write it in this form to make more apparent the analogy between MCF and minimal surfaces.)

The monotonicity formula says that $\Theta(\mathcal{M}, X, r)$ is increasing in $r = |t - t_0|^{1/2}$, and is strictly increasing unless we are a self-shrinker. Keep in mind that as r increases, we look backwards in time! The Gaussian density at X is therefore the well-defined limit

$$\Theta(\mathcal{M}, X) := \lim_{r\to 0} \Theta(\mathcal{M}, X, r).$$

If $\mathcal{M}$ is an ancient flow, the density at ∞ is also defined:

$$\Theta(\mathcal{M}) := \lim_{r\to\infty} \Theta(\mathcal{M}, X, r),$$

which one can easily show is independent of X.

Let's take a moment to interpret the Gaussian density itself, or the "F-functional" as Colding-Minicozzi call it.

Proposition 10.2. *Provided $\mu B_r(0)$ grows sub-exponentially, then F_m is the weighted average of Euclidean area densities. Precisely,*

$$F(\mu) = \int_0^\infty \frac{\omega_m r^{m+2} e^{-r^2/4}}{2(4\pi)^{m/2}} \frac{\mu B_r(0)}{\omega_m r^m} dr.$$

Proof. Writing $A(r) = \mu B_r$, we have

$$\begin{aligned} F(\mu) &= \frac{1}{(4\pi)^{m/2}} \int_{R^N} e^{-|x|^2/4} d\mu(x) \\ &= \frac{1}{(4\pi)^{m/2}} \int_{r=0}^{\infty} e^{-r^2/4} dA(r) \\ &= \frac{1}{(4\pi)^{m/2}} \int_0^{\infty} d(e^{-r^2/4}A) + \frac{1}{(4\pi)^{m/2}} \int_0^{\infty} \frac{r}{2} e^{-r^2/4} A dr \\ &= \frac{1}{(4\pi)^{m/2}} \int_0^{\infty} \omega_m \frac{r^{m+1}}{2} e^{-r^2/4} \frac{A(r)}{\omega_m r^m} dr \end{aligned}$$

One can verify that the total mass of the weighted integral is 1. □

Notice the weighting factor drops exponentially. We see that the Gaussian and Euclidean densities are comparable.

Corollary 10.3. *If $\mu B_r(0)$ grows sub-exponentially, then for every R there is a constant $C = C(m, R)$ such that*

$$C \frac{\mu B_R(0)}{\omega_m R^M} \leq F_m(\mu) \leq \sup_r \frac{\mu B_r(0)}{\omega_m r^m}.$$

Definition 10.3.1. The m-entropy of manifold M (or Radon measure μ) is defined to be

$$E_m(M) = \sup_{\lambda > 0, p \in R^N} F_m(p + \lambda M).$$

By blowing up or down at a given point, and recalling that F is the (exponentially weighted) average of Euclidean densities, we have

Proposition 10.4. *Given any Radon μ, we have*

$$E_m(\mu) \geq \Theta^m_{eucl}(\mu, x) := \lim_{r \to 0} \frac{\mu B_r(x)}{\omega_m r^m},$$

and

$$E_m(\mu) \geq \Theta^m_{eucl}(\mu) := \lim_{r \to \infty} \frac{\mu B_r(x)}{\omega_m r^m}$$

whenever these limits exists.

Since we can write

$$E(M) = \sup_{x_0, r > 0} \frac{1}{(4\pi r^2)^{m/2}} \int_M e^{-|x - x_0|^2/4r^2},$$

if $\mu(t)$ is a Brakke flow then by appropriately recentering $X = (x_0, t_0)$ in the monotonicity formula we see that $E(\mu(t))$ is decreasing in time.

In particular, the set

$$\mathcal{C}_\Lambda = \{\text{Radon measures with entropy } E_m(\mu) \leq \Lambda\}$$

is preserved by Brakke flow. Since entropy (and hence Gaussian density) bounds give local area bounds, the space of flows in $\mathcal{C}_\Lambda$ is compact.

Also, because E is decreasing, we can define a notion of "entropy of an ancient flow", which agrees with the density at infinity:

$$E(\mathcal{M}) := \lim_{t\to-\infty} E(M_t) = \sup_{r,X} \Theta(\mathcal{M}, X, r) = \Theta(\mathcal{M}).$$

We establish elementary facts about densities and limit flows.

Proposition 10.5. *Suppose $\mathcal{M}_i \rightharpoonup \mathcal{M}$ as Brakke flows, $X_i \to 0$, and $r_i \to 0$. Then*

$$\limsup_i \Theta(\mathcal{M}_i, X_i, r_i) \leq \Theta(\mathcal{M}, X).$$

In other words, Guasian density is upper-semicontinuous.

Proof. By replacing $\mathcal{M}_i$ with $\mathcal{M}_i - X_i$ we can assume $X_i = X = (0,0)$. Since the associated measures converge for every t, we have

$$\Theta(\mathcal{M}_i, 0, r_i) \leq \Theta(\mathcal{M}_i, 0, r) \to \Theta(\mathcal{M}, 0, r).$$

Therefore $\limsup_i \Theta(\mathcal{M}_i, 0, r_i) \leq \Theta(\mathcal{M}, 0, r)$. Now take $r \to 0$. □

Theorem 10.6 (existence of limit flows)**.** *Let $\mathcal{M}_i \rightharpoonup \mathcal{M}$, $X_i \to X$ and $\lambda_i \to \infty$. Then there is a subsequence i', and an eternal Brakke flow $\tilde{\mathcal{M}}$, such that*

$$\mathcal{D}_{\lambda_{i'}}(\mathcal{M}_i - X_i) \rightharpoonup \tilde{\mathcal{M}}$$

and $\Theta(\tilde{\mathcal{M}}) \leq \Theta(\mathcal{M}, X)$.

Proof. The existence of a subsequence and limit $\tilde{\mathcal{M}}$ follows directly from the compactness theorem, since entropies are uniformlyed bounded. Since $E(\tilde{M}) = \sup_r \Theta(\mathcal{M}, X, r)$, the entropy bound follows directly from the upper semi-continuity proposition. □

Remark 10.7. The same theorem holds true for flows only defined in a region, by using an appropriate cut-off function to localize entropy. For example, letting

$$f = \frac{1}{R^2}(R^2 - |x|^2 - 2mt)_+,$$

we have that $\mu(t)(f\rho)$ is decreasing for (short) time.

Proposition 10.8. *For any non-zero ancient flow, $\Theta(\mathcal{M}) \geq 1$.*

Proof. For a.e. $t < T_{extinction}$, $\mu(t)$ is an integral varifold. In particular, for $\mu(t)$-a.e. x, we have $\Theta^m_{eucl}(\mu(t), x) \geq 1$. Therefore

$$E(\mu(t)) \geq \Theta_{eucl}(\mu(t), x) \geq 1,$$

and hence $\Theta(\mathcal{M}) = \sup_t E(\mu(t)) \geq 1$. □

Corollary 10.9. *If $\mathcal{M}$ is a Brakke flow, than $\Theta(\mathcal{M}, X) \geq 1$ at all points $X \in \mathrm{spt}\mathcal{M}$.*

Proof. Choose $X_i = (x_i, t_i)$ such that $\theta_e(\mu(t_i), x_i) \in Z_+$. By considering the tangent flow at each X_i, and using the previous proposition, we have that

$$\Theta(\mathcal{M}, X_i) \geq \Theta^m_{eucl}(\mu(t_i), x_i) \geq 1.$$

Now use the upper semicontinuity of Θ. □

11 Brakke regularity

There is a very strong analogy between Allard's regularity theorem and Brakke's regularity theorem. Allard says that if M is a minimal variety with $1 \leq \theta_e(M, x) \leq 1 + \epsilon$ (for ϵ depending only on the dim and codim), then x is a regular point.

Brakke's theorem will say that if $\mathcal{M}$ is an integral Brakke flow, and $1 \leq \Theta(\mathcal{M}, X) \leq 1 + \epsilon$, then X is a regular point.

The full regularity theorem for integral Brakke flows is very involved, but we will prove an "easy" Brakke regularity theorem for smooth flows. This will in fact suffice to prove Brakke's full regularity theorem for flows obtained via elliptic regularization, since these flows have the special property of being a limit of smooth flows.

Theorem 11.1 (easy Brakke I)**.** *Let $\mathcal{M}_i$ be smooth Brakke flows, with $\mathcal{M}_i \rightharpoonup \mathcal{M}$. If $\Theta(\mathcal{M}, X) = 1$, then there is a spacetime neighborhood of X such that*

$$\mathcal{M}_i \cap U \to \mathcal{M} \cap U \text{ smoothly.}$$

Proof. Let $\kappa(\mathcal{M}_i, Y)$ be the norm of the second fundamental form of $\mathcal{M}_i$ at Y. It suffices to prove that $\kappa(\mathcal{M}_i, \cdot)$ is uniformly bounded in some U.

Suppose the contrary: there is a sequence $X_i \to X = 0$ with $\kappa(\mathcal{M}_i, X_i) \to \infty$. Choose a sequence R_i such that: a) $R_i \geq 2|X_i|$, b) $R_i \to 0$, and c) $R_i\kappa(\mathcal{M}_i, X_i) \to \infty$. Here we use the parabolic norm

$$|(x, t)| = \max\{|x|, |t|^{1/2}\}.$$

One could of course alternatively use $|(x, t)| = \sqrt{|x|^2 + |t|}$, or any other parabolic-scale-invariant norm.

Let U_i be the R_i-parabolic-neighborhood of 0, and let Y_i maximize the parabolic-scale-invariant quantity

$$\kappa(\mathcal{M}_i, \cdot)\mathrm{dist}(\cdot, \partial U_i).$$

By our choice of R_i, we have $Y_i \to 0$ (from b), and $\kappa(\mathcal{M}_i, Y_i)\mathrm{dist}(Y_i, \partial U_i) \to \infty$ (from a and c).

Set

$$\tilde{\mathcal{M}}_i = \kappa(\mathcal{M}_i, Y_i)(\mathcal{M}_i - Y_i), \quad \tilde{U}_i = \kappa(\mathcal{M}_i, Y_i)(U_i - Y_i).$$

We have

$$\begin{aligned}\mathrm{dist}(0, \partial\tilde{U}_i) &= \kappa(\tilde{\mathcal{M}}_i, 0)\mathrm{dist}(0, \partial\tilde{U}_i) \\ &= \max_{\tilde{\mathcal{M}}_i \cap \tilde{U}_i} \kappa(\tilde{\mathcal{M}}_i, \cdot)\mathrm{dist}(\cdot, \partial\tilde{U}_i) \to \infty,\end{aligned}$$

so $\tilde{U}_i \to R^N \times R$. Further,

$$\kappa(\tilde{\mathcal{M}}_i, Y) \leq \frac{\mathrm{dist}(0, \partial\tilde{U}_i)}{\mathrm{dist}(Y, \partial\tilde{U}_i)} \to 1 \text{ for } Y \text{ in a bounded region.}$$

So $\kappa(\tilde{\mathcal{M}}_i, \cdot)$ is uniformly bounded on compact subsets of spacetime.

We deduce that $\tilde{\mathcal{M}}_{i'} \to \tilde{\mathcal{M}}$ smoothly on compact subsets of $R^N \times R$, for some subsequence i' and some smooth flow $\tilde{\mathcal{M}}$. By our normalization we have

$$\kappa(\tilde{\mathcal{M}}, \cdot) \leq \kappa(\tilde{\mathcal{M}}, 0) = 1.$$

By the upper-semi-continuity proposition $\Theta(\tilde{\mathcal{M}}) \leq \Theta(\mathcal{M}, X) = 1$. But $0 \in \tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}$ is smooth, so by monotonicity

$$\Theta(\tilde{\mathcal{M}}) = \Theta(\tilde{\mathcal{M}}, 0, \infty) \geq \Theta(\tilde{\mathcal{M}}, 0) \geq 1.$$

So in fact $\Theta(\tilde{\mathcal{M}}, 0, \cdot)$ is constant. Therefore $\tilde{\mathcal{M}}$ is a parabolic cone, which is regular at the origin, and hence $\tilde{\mathcal{M}}$ is flat. But this implies $\kappa(\tilde{\mathcal{M}}, 0) = 0$, a contradiction. □

Remark 11.2. One could of course take κ to be a more obvious "spacetime norm" like

$$\kappa = |A|_{slice}| + |velocity|,$$

but in our case $|velocity| \leq |A|_{slice}|$ anyway.

We've only shown the regularity theorem with $\epsilon = 0$. The following "gap theorem" gives us the full Brakke's theorem.

Theorem 11.3 (gap theorem)**.** *Suppose $\mathcal{M}$ is a self-similar Brakke flow, which isn't a multiplicity 1 plane. Then $\Theta(\mathcal{M}) \geq \eta > 1$ for some $\eta = \eta(dim, codim)$.*

Proof. Otheriwse, we can find a sequence $\mathcal{M}_i$ of self-shrinkers with $\Theta(\mathcal{M}_i) \to 1$.

For each $p \in \mathcal{M}_i(-1)$, we have

$$\Theta_{eucl}(\mathcal{M}_i(-1), p) \leq \Theta(\mathcal{M}_i, (p, -1)) \leq \Theta(\mathcal{M}_i) \to 1.$$

Since entropies are unformly bounded, we can pass to a subsequence and obtain $\mathcal{M}_i \rightharpoonup \tilde{\mathcal{M}}$ as Brakke flows, for some (self-similar) Brakke flow $\tilde{\mathcal{M}}$. We have $\Theta(\tilde{\mathcal{M}}) = 1$ and $\theta_e(\tilde{\mathcal{M}}, \cdot) \leq \Theta(\tilde{\mathcal{M}}, \cdot) \leq 1$.

Recall that self-shrinkers are minimal for a weighted metric (theorem REF). So by Allard's theorem $\tilde{\mathcal{M}}(-1)$ is smooth everywhere, and hence the convergence $\mathcal{M}_i(-1) \to \tilde{\mathcal{M}}(-1)$ is smooth also. So $\mathcal{M}_i$ is smooth for $t < 0$.

But since $\mathcal{M}_i$ are not flat, we must have a sequence (x_i, t_i) converging $|(x_i, t_i)| \to 0$ and satsifying $\kappa(\mathcal{M}_i, (x_i, t_i)) \to \infty$. Apply "easy Brakke" to $\mathcal{M}_i \cap \{t \leq t_i\}$, knowing that

$$\mathcal{M}_i \cap \{t \leq t_i\} \rightharpoonup \tilde{\mathcal{M}}\{t \leq 0\} \text{ and } \Theta(\tilde{M}) = \Theta(\tilde{M}, 0) = 1.$$

We deduce the convergence is smooth, a contradiction. □

Theorem 11.4. *Let*

$$\mathcal{D} = \left\{ \begin{array}{c} \textit{integral Brakke flows } \mathcal{M} \textit{ such that every } X \\ \textit{with } \Theta(\mathcal{M}, X) = 1 \textit{ is a regular point} \end{array} \right\}.$$

Then $\mathcal{D}$ is closed, and if $\mathcal{M} \in \mathcal{D}$ satsifies $\Theta(\mathcal{M}, X) < \eta$ (with η as in the gap theorem), then X is a regular point.

Corollary 11.5. *Limits of smooth, embedded flows are in $\mathcal{D}$. In particular, flows constructed by elliptic regularization are in $\mathcal{D}$.*

Proof of theorem. Suppose $\mathcal{M}_i \rightharpoonup \mathcal{M}$ with $\mathcal{M}_i \in \mathcal{D}$, and $\Theta(\mathcal{M}, X) < \eta - \epsilon$. The gap theorem implies in fact $\Theta(\mathcal{M}, X) = 1$.

The upper semi-continuity implies that for i sufficiently large, $\Theta(\mathcal{M}_i, \cdot) \leq \eta - \epsilon$ in some fixed neighborhood U. By the gap theorem $\Theta(\mathcal{M}_i, \cdot) \equiv 1$ on U, and so by our assumption $\mathcal{M}_i \cap U$ is regular.

Now apply easy Brakke to $\mathcal{M}_i \cap U \rightharpoonup \mathcal{M} \cap U$. □

12 stratification

Almgren and Brian's stratification (based on Federer's dimension reducing argument) is the key step to proving partial regularity. The intuition is the following: if every tangent cone in a subset is at most k-dimensional, then the subset is at most k-Hausdorff-dimensional. The technique is very general, extending to minimal varieties, harmonic maps, mean curvature flows, limits of manifolds with $Ric \geq -C$. The main tool required is some kind of monotonicity.

Precisely, we interpret the dimensionality of a (tangent) cone C using the notion of "spine", which is the biggest subspace R^k so that $C = R^k \times C_0$. We stratify the minimal surface or mean curvature flow by the maximum spine of any tangent cone. A dimension reducing argument shows that the k-strata (points whose tangent cone has spine at most k-dimensional) is at most k-Hausdorff dimensional.

We then play the following game. By ruling out low-dimensional (non-trivial) tangent cones in sing$M$, we rule out high-dimensional spines, and thereby rule out high Hausdorff dimension.

We first motivate our construction by reviewing Almgren's stratifcation for minimal varieties.

Lemma 12.1. *If* $0 \in M \subset R^N$*, set*

$$V(M) = \{x \in R^N : x + M = M\}.$$

Of course $V(M)$ *is an additive subgroup. If* M *is a cone, then* $V(M)$ *is a subspace, and* $M = C \times V(M)$*, for* $C = M \cap V(M)^\perp$*.*

Theorem 12.2. *If* $M \subset R^N$ *is a minimal cone, then* $\max \Theta(M, \cdot) = \Theta(M, 0) = \Theta(M)$*. If we define*

$$\mathrm{spine}(M) = \{x : \Theta(M, x) = \Theta(M)\},$$

then $\mathrm{spine}(M) = V(M)$*. In particular, we have*

$$M = C \times \mathrm{spine}(M), \quad \textit{for } C = M \cap (\mathrm{spine} M)^\perp.$$

Proof. We have $\Theta(M, x) \leq \Theta(M)$, with equality iff M is a cone over x. Trivially $V(M) \subset \mathrm{spine}(M)$. Conversely, if $a \in \mathrm{spine}(M)$, then M is a cone over both 0 and a. In other words, for any λ

$$\lambda M = M = a + \lambda(M - a),$$

and so $M = \frac{1-\lambda}{\lambda} a + M$. Set $\lambda = 1/2$. □

Theorem 12.3 (Almgren's stratification)**.** *Let* M *be a minimal variety. Define the* k*-stratum*

$$\begin{aligned} \mathcal{S}_k(M) &= \{x \in M : \textit{every tangent cone at } x \textit{ has } \dim \mathrm{spine} \leq k\} \\ &= \{x \in M : \textit{every tangent cone at } x \textit{ has at most} \\ &\qquad k\textit{-dims of translation symmetry}\}. \end{aligned}$$

Then $\dim \mathcal{S}_k(M) \leq k$*.*

Remark 12.4. Jean Taylor proved the stronger result that if tangent cones look like $Y \times R^k$, where Y is the stable Y-configuration of half-lines in R^2, then the surface also locally looks like $Y \times R^k$. I.e. so locally M looks like three $(k+1)$-dimensional submanifolds meeting along a $C^{1,\alpha}$ line of intersection. Leon Simon proved a similar result in more generality.

This works because any perturbation of the Y must still have a singularity, which isn't true for minimizing hypercones because of the Hardt-Simon foliation.

Example 12.5. If M^m is minimizing mod-2, then multiplicity is not an issue (everything is multplicity one). So

$$\mathcal{S}_m(M) = \{\text{multiplicity-1 planes}\} = \mathrm{reg} M,$$

so $\mathrm{sing} M \subset \mathcal{S}_{m-1}(M)$.

But if C is a 1-dimensional mod-2 minimizer, then C is a (multiplicity 1) line. So in fact

$$\mathcal{S}_{m-1}(M) = \mathcal{S}_m(M) = \mathrm{reg} M$$

and thus $\mathrm{sing} M \subset \mathcal{S}_{m-2}(M)$, yielding the bound $\dim_{\mathcal{H}} \mathrm{sing} M \leq m - 2$.

Now consider an m-dimensional mean curvature flow $\mathcal{M}$.

Definition 12.5.1. A tangent flow at $X = (x, t)$ is any subsequential limit

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - X) \rightharpoonup \mathcal{M}'.$$

Of course $\mathcal{M}'$ is eternal, but since we only have a backwards monotonicity we can only say something about negative times.

Theorem 12.6. *$\mathcal{M}' \cap \{t < 0\}$ is $\mathcal{D}_\lambda$-invariant. More generally, if $\mathcal{M}'$ is any eternal flow with $\Theta(\mathcal{M}', 0) = \Theta(\mathcal{M}')$, then $\mathcal{M}' \cap \{t < 0\}$ is $\mathcal{D}_\lambda$-invariant.*

Here's a question: could one show that $\mathcal{M}' \cap \{t > 0\}$ is necessarily a self-expander? Is there any kind of uniqueness of extensions?

Remark 12.7. Colding-Ilmanen-Minicozzi proved that if one tangent flow is a shrinking cylinder $S^k \times R^{m-k}$, then every tangent flow is a cylinder. One can calculate that the densities of each cylinder $S^k \times R^{m-k}$ is distinct, so this implies every tangent cone is some rotation of the same cylinder. Colding-Minicozzi later proved that no rotations are possible, i.e. if one tangent flow is a cylinder, all tangent flows are precisely the same cylinder.

Definition 12.7.1. Given a Brakke flow $\mathcal{M}$, define the spatial spine to be the subspace

$$\begin{aligned} V(\mathcal{M}) &= \{x \in R^N : \Theta(\mathcal{M}, (x, 0)) = \Theta(\mathcal{M})\} \\ &= \{x \in R^N : \mathcal{M} \cap \{t < 0\} \text{ is invariant under translation by } (x, 0)\}. \end{aligned}$$

We have the following possibilities for the "spacetime spine":

$$\{X : \Theta(\mathcal{M}, X) = \Theta(\mathcal{M})\} = \begin{cases} V(\mathcal{M}) \times \{0\} & \text{"actual shrinker", e.g. cylinder} \\ V(\mathcal{M}) \times R & \text{"static cone", e.g. minimal cone} \\ V(\mathcal{M}) \times (-\infty, a] & \text{"quasistatic cone"} \end{cases}$$

Quasistatic cones (possibility 3) can occur as the tangent flow for the cusp of an immersed flow of curves. PICTURE. As a limit of Grim-Reaper-like curves, we would have

$$\text{tangent flow} = (\text{multiplicity 2 line}) \times (-\infty, 0].$$

Definition 12.7.2. Let $\mathcal{M}'$ be a tangent flow. Let $d(\mathcal{M}') = \dim V(\mathcal{M}')$ be the dimension of the spatial spine. Define the parabolic dimension of $\mathcal{M}'$ to be

$$D(\mathcal{M}') = \begin{cases} d + 2 & \text{if } \mathcal{M}' \text{ is a static cone (possibility 2)} \\ d & \text{if } \mathcal{M}' \text{ is quasistatic or not static (possibilities 1 or 3)} \end{cases}.$$

Theorem 12.8. *Let $\mathcal{M}$ be a Brakke flow. Define*

$$\mathcal{S}_k(\mathcal{M}) = \{X : D(\mathcal{M}') \le k \textit{ for every tangent flow of } \mathcal{M}' \textit{ at } X\}.$$

Then the parabolic Hausdorff dimension of $\mathcal{S}_k(\mathcal{M})$ is $\le k$. By parabolic Hausdorff we mean the Hausdorff measure associated to the metric

$$d(X, X') = |x - x'| + |t - t'|^{1/2}.$$

Proof. We can adapt directly Federer's dimension reducing argument, provided we work with a quantization of the strata. Let us say $\mathcal{M}\llcorner B^{sp}_R(X)$ is (k,ϵ)-symmetric if $\mathcal{M}\cap B_R(X)$ is ϵ-close (in a scale-invariant notion of varifold distance) to a parabolic cone $\mathcal{M}'$ with $D(\mathcal{M}')\geq k$.

Now define the effective strata to be

$$\mathcal{S}^k_\epsilon(\mathcal{M})=\{X:\mathcal{M}\llcorner B_R(X) \text{ is *not* } (\epsilon,k+1)\text{-symmetric, } \forall R\in(0,1]\}.$$

One can readily verify that $\mathcal{S}^k=\cup_\epsilon\mathcal{S}^k_\epsilon$, and that (crucially) the $S^k_\epsilon(\mathcal{M})$ are closed under varifold convergence.

Suppose, towards a contradiction, that $\mathcal{H}^\alpha_{par}(\mathcal{S}^k(\mathcal{M}))>0$. Then for some $\epsilon>0$, we have $\mathcal{H}^\alpha_{par}(\mathcal{S}^k_\epsilon(\mathcal{M}))>0$ also.

By the usual density argument, and because $\mathcal{H}^\alpha_{par}$ scales parabolically, we can therefore find a point X_1, and a tangent flow $\mathcal{M}_1$ at $X_1\in\mathcal{S}^k_\epsilon(\mathcal{M})$, so that $\mathcal{H}^\alpha_{par}(\mathcal{S}^k_\epsilon(\mathcal{M}_1))>0$. By definition we have $\alpha>D(\mathcal{M}_1)$.

Let us consider two cases. First, suppose $\mathcal{M}_1$ is static. Then since $\dim V(\mathcal{M}_1)+2<\alpha$, we have $\mathcal{H}^\alpha_{par}(\mathcal{S}^k_\epsilon(\mathcal{M}_1)\sim V(\mathcal{M}_1)\times R)>0$. We can therefore repeat the previous process with some

$$X_2\in S^k_\epsilon(\mathcal{M}_1)\sim V(\mathcal{M}_1)\times R,$$

and obtain a static tangent flow $\mathcal{M}_2$ with a spatial spine of greater dimension. So, $D(\mathcal{M}_2)\geq D(\mathcal{M}_1)+1$.

Suppose now $\mathcal{M}_1$ is q-static, or non-static. Then $\mathcal{H}^\alpha_{par}(\mathcal{S}^k_\epsilon(\mathcal{M}_1)\sim V(\mathcal{M}_1)\times\{0\})>0$, and so we can blow up at some $X_2\in\mathcal{S}^k_\epsilon(\mathcal{M}_1)\sim V(\mathcal{M}_1)\times\{0\}$ to obtain an $\mathcal{M}_2$.

Write $X_2=(x_2,t_2)$. If $t_2=0$, then $\mathcal{M}_2$ is either q-static or non-static also, and $V(\mathcal{M}_2)\geq V(\mathcal{M}_1)+1$. If $t_2<0$, then $\mathcal{M}_2$ is now static, and so $D(\mathcal{M}_2)\geq D(\mathcal{M}_1)+2$.

In all cases, we obtain a new parabolic cone $\mathcal{M}_2$, with $\mathcal{H}^\alpha_{par}(S^k_\epsilon(\mathcal{M}_2))>0$, but $D(\mathcal{M}_2)\geq D(\mathcal{M}_1)+1$. If $\alpha>k$, we can continue this process until we arrive at a contradiction. Therefore we must have

$$\mathcal{H}^\alpha_{par}(\mathcal{S}^k_\epsilon(\mathcal{M}))=0\quad\forall\alpha>0.$$

□

Corollary 12.9. *For a.e. t, $\dim(\mathcal{S}_k\cap\{time=t\})\leq k-2$.*

Example 12.10. PICTURE

We enumerate some possible high-dimensional tangnet cones. Take $\mathcal{M}$ to be a flow of m-dimensional surfaces in R^N. Of course $d\leq m$, and so $D\leq m+2$.

d	$\mathcal{M}\cap\{t<0\}$	D
m	m-plane (w/ multiplicity)	$m+2$ (static)
		m (quasi-static)
$m-1$	$R^{m-1}\times$ (1-d minimal cone)	$m+1$ (static)
		$m-1$ (q-static)
	$R^{m-1}\times$ (1-d shrinker)	$m-1$
$m-2$	$R^{m-2}\times$ (2-d minimal cone)	m (static)
		$m-2$ (q-static)
	$R^{m-2}\times$ (2-d shrinker)	$m-2$

Remark 12.11 (key idea)**.** If we can rule out static planes of multiplicity > 1, then $D \leq m+1$. If we can rule out static configurations of half-planes, then $D \leq m$. If we can rule out static cones $R^{m-2} \times$ (2-d minimal cone), and quasi-static planes, then $D \leq m-1$.

Theorem 12.12 (easy parity theorem)**.** *Let $\mathcal{G}$ be the set of Brakke flows $\mathcal{M} \in \mathcal{D}$ satisyfing the following for every small closed curve C in spacetime:*

$$C \cap \text{sing}\mathcal{M} = \emptyset, \ C \textit{ hits } \text{reg}\mathcal{M} \textit{ transversely} \implies \#(C \cap \text{reg}\mathcal{M}) \textit{ is even.}$$

Then $\mathcal{G}$ is closed. (Here regular means smooth and multiplicity one.)

Corollary 12.13. *Limits of smooth, embedded flows are in $\mathcal{G}$. In particular, flows constructed via elliptic regularization are in $\mathcal{G}$.*

Proof of theorem. Follows directly from the fact that convergence is smooth near regular points of the limit. □

Theorem 12.14. *Let $\mathcal{M}$ be an m-dimensional Brakke flow in $\mathcal{G}$. Let W be the open set*

$$W = \{X : \Theta(\mathcal{M}, X) < 2\}.$$

Then $\text{sing}\mathcal{M} \cap W$ *has* $\dim_{par} \leq m-1$.

Proof. Following remark KEYIDEA, we simply need to rule out various tangent flows. Since we require $\Theta < 2$, every (quasi)-static m-plane is multiplicity 1. Since $\mathcal{M} \in \mathcal{D}$, density 1 points are regular. This shows $D \leq m+1$.

Suppose we had a non-planar (q-)static union of half-planes. The density hypothesis implies there must be 3 half-planes, meeting at 120 degrees. This violates the easy parity theorem. So $D \leq m$.

Suppose we had a static $R^{m-2} \times$ (2-d cone). Then cone $\cap\, \partial B_1$ is a geodesic network in the sphere. The density hypothesis and easy parity theorem rule out junctions (the tangent cones at these points would be unions of half-planes). So

$$\begin{aligned} \text{cone} \cap \partial B_1 &= \text{union of closed geodesics} \\ &= \text{single great circle} \end{aligned}$$

Thus we are at a regular point. This shows $D \leq m-1$. □

Remark 12.15. If we have a non-planar tangent flow $R^{m-1} \times$ (1-d shrinker), then the density hypothesis and parity theorem imply the shrinker must be smooth and embedded, and hence must be S^1. So if we could rule out tangent flows of the form $R^{m-1} \times S^1$ and $R^{m-3} \times$ (3-d shrinker), we could deduce $D \leq m-2$ in W.

In the study of mean-convex MCF static flows of density ≤ 2 are of crucial importance. The previous theorem gives good regularity for these kinds of limits.

Corollary 12.16. *Let $\mathcal{M}$ be a (q-)static M in $\mathcal{G}$. If $\Theta(\mathcal{M}) \le 2$, and M is not a mult-2 plane, then*

$$\dim \operatorname{sing} M \le m - 3$$

Proof. Since $\mathcal{M}$ is not a static, mult-2 plane, we have $W = \mathcal{M}$. Therefore $\dim_{par} \operatorname{sing}\mathcal{M} \le m-1$, but the time axis as parabolic dimension 2, and therefore $\dim \operatorname{sing} M \le m - 3$. □

13 barrier and maximum principles

We begin with an elementary maximum principle.

Theorem 13.1. *Let $t \in (a, b] \mapsto \mu(t)$ be an m-dim rectifiable or integral Brakke flow in $\Omega \subset R^N$, and let M denote its spacetime support. Let $u : \Omega \times (a, b] \to R$ be a function satisfying*

$$\partial_t u - \operatorname{tr}_m D^2 u < 0 \quad \text{at } (x_0, t_0).$$

Here $\operatorname{tr}_m D^2 u = \sum_{i=1}^m \lambda_i$, where $\lambda_1 \le \ldots \le \lambda_N$ are the eigenvalues of $D^2 u$.

Then $u|_{M \cap \{t \le t_0\}}$ cannot have a local maximum at (x_0, t_0).

Proof. WLOG take $(x_0, t_0) = 0$ and $\mathcal{M} = \mathcal{M} \cap \{t \le 0\}$. We can also suppose $u|_{\mathcal{M}}$ has a strict lcoal maximum at $(0, 0)$, for otherwise we can replace u by $u - \epsilon(|x|^2 + |t|^4)$.

Let $Q(r) = B_r(0) \times (-r^2, 0]$, where we choose r sufficiently small so that: a) $-r^2 \ge t_{initial}$; b) $u|_{M \cap \bar{Q}}$ has a maximum only at $(0, 0)$; c) $\partial_t u - \operatorname{tr}_m D^2 u < 0$ on $\bar{Q}$; d) $u|_{M \cap (\bar{Q} \sim Q)} < 0 < u|_{(0,0)}$. To achieve d) we may need to add a constant to u.

Plug u_+ (or u_+^4) into the Brakke flow inequality. Notice that $u_+|_{\mathcal{M}}$ is supported in a compact subset of $\bar{Q}$.

We then have the contradiction

$$\begin{aligned}
0 &= \int u_+ d\mu(0) \\
&\le \int u_+ d\mu(0) - \int u_+ d\mu(-r^2) \\
&\le \int_{-r^2}^0 \int -|H|^2 u_+ + H \cdot Du_+ + \partial_t u_+ d\mu(t) dt \\
&\le \int_{-r^2}^0 \int -div(Du_+) + \partial_t u_+ d\mu(t) dt \\
&\le \int_{-r^2}^0 \int -\operatorname{tr}_m D^2 u_+ + \partial_t u_+ \\
&< 0.
\end{aligned}$$

□

Remark 13.2. If we instead assume only $\partial_t - \operatorname{tr}_m D^2 u \le 0$ at (x_0, t_0), then we can deduce u has no strict local maximum at (x_0, t_0).

Consider a 1-paramater family of domains $t \in (a, b] \mapsto N(t)$ in Ω, so that $t \mapsto \partial N(t)$ is a smooth 1-parameter family of hypersurfaces. Write $M(t) = \{x : (x, t) \in M\}$ for the t-timeslice of M. We prove first a weak maximum principle for N. Notice the codimension of the Brakke flow $\mu(t)$ can be arbitrary.

Theorem 13.3. *Suppose $M(t) \subset N(t)$ for all times. If for some τ we have $p \in M(\tau) \cap \partial N(\tau)$, then necessarily*

$$v(p, \tau) \geq h_m(p, \tau),$$

*where v is the inward velocity of ∂N, and $h_m = \kappa_1 + \ldots + \kappa_{N-1}$, for $\kappa_1 \leq \ldots \leq \kappa_{N-1}$ the principle curvatures of ∂N with respect to the *inward* unit normal.*

Proof. Let $u : \Omega \times (a, b] \to R$ be the signed distance funtion to $\partial N(t)$, i.e.

$$u(x,t) = \begin{cases} -\mathrm{dist}(x, \partial N(t)) & x \in N(t) \\ \mathrm{dist}(x, \partial N(t)) & x \notin N(t) \end{cases}.$$

Let $e_1, \ldots, e_{N-1}$ be the principle direction of $\partial N(t)$ at p. In this basis, $D^2u|_p$ takes the form

$$D^2u|_p = \begin{pmatrix} \kappa_1 & & & \\ & \ddots & & \\ & & \kappa_{N-1} & \\ & & & 0 \end{pmatrix}.$$

So $\mathrm{tr}_m D^2u(p, \tau) = h_m(p, \tau)$, and of course $\partial_t u(p, \tau) = v(p, \tau)$. Now apply the previous maximum principle. □

Remark 13.4. A strong maximum principle is true for codimension-1 Brakke flows (c.f. Solomon-White's result for minimal varieities). A similar thing should be true for higher codimensions but I don't know of any results.

We prove a weak and a strong barrier principle. Suppose now $m+1 = N$, so $\mu(t)$ is a codimension-1 Brakke flow in R^{m+1}, and M is its spacetime support.

Theorem 13.5 (weak version)**.** *If $M(t) \subset N(t)$ for all time, and $p \in M(t) \cap \partial N(t)$, then*

$$(\textit{velocity of } M(t) \textit{ at } p) \cdot \nu_{\partial N, inner} \geq H_{\partial N} \cdot \nu_{\partial N, inner}.$$

Proof. Just a restatement of the weak maximum principle Theorem REF. □

Theorem 13.6 (strong version)**.** *Assume $\partial N(t)$ is compact, and that*

$$v_{\partial N, inner} \leq H_{\partial N, inner}$$

everywhere. Suppose $M(0) \subset N(0)$, and $\partial N(0) \sim M(0)$ is non-empty.

Then $M(t) \subset \mathrm{int} N(t)$ for $t > 0$. I.e. "$M(t)$ and $\partial N(t)$ become disjoint immediately"

We prove the strong barrier principle when $t \mapsto \partial N(t)$ moves by MCF.

Lemma 13.7. *If $v_{\partial N,inner} \leq H_{\partial N,inner}$ and $M(0) \subset N(0)$, then $M(t) \subset N(t)$ for all time.*

Proof of lemma. Let $N_\epsilon(t)$ be the flow such that $\partial N_\epsilon(0) = \partial N(0)$, and $vel_\epsilon = vel - \epsilon$. By the weak barrier principle $M(t) \subset N_\epsilon(t)$. Now take ϵ to 0. □

Proof of Theorem. Let $\hat{N}(0)$ be a smooth domain such $M(0) \subset \hat{N}(0) \subsetneq N(0)$. PICTURE. Flow $t \mapsto \partial\hat{N}(t)$ by MCF. By the classical maximum principle, $\hat{\partial}N(t)$ and $\partial N(t)$ become disjoint immediately. Now apply the previous lemma. □

14 mean convex flows

We study mean convex mean curvature flow. The following is a simplification of the results in Brian's two papers (size and nature of singular set). There may be some complications if in a Riemannian manifold if $n > 6$, or past the first singular time.

Let $M(0)^n = \partial K(0)$ be a smooth, compact, mean-convex region in R^{n+1}. We can obtain a Brakke flow $\mu(t)$ with $\mu(0) = \mu_{M(0)}$ via elliptic regularization. In fact by mean-convexity we can minimize $\mathcal{I}_\lambda$ in the region $K(0) \times R$, and obtain M_λ as a graph over $K(0) \times \{0\}$. So the translators $t \mapsto M_\lambda - \lambda t$ are nested, and hence the limit is nested also.

Ilmanen showed (REF) that the level set flow starting at such an $M(0)$ is non-fattening, so up to sudden vanishing the Brakke flow $\mathcal{M}$ starting at $M(0)$ is unique. Therefore we can assume $\mathcal{M}$ arises from the elliptic regularization process, and *in particular*, is a limit of smooth flows.

A further consequence of non-fattening is that we can equivocate between the level-set and Brakke formulations of the flow. Write $K(t)$ for the level-set evolution of the set $K(0)$, and $\mathcal{K}$ for its spacetime track. If $M(t)$ is the non-vanshing Brakke flow starting at $M(0)$, then $M(t) = \partial K(t)$.

Important note! The mean-convex MCF as coinciding with the level set flow is *strictly* nested. Any limit is only *weakly* nested. In fact in general the level set flow is not closed under limits.

We work towards proving the following partial regularity result for $\mathcal{M}$:

Theorem 14.1. *Let $\mathcal{M}$ be the Brakke flow of $M(0)^n$ as above. If $\mathrm{sing}\mathcal{M}$ is the spacetime singular set of $\mathcal{M}$, then $\mathrm{sing}\mathcal{M}$ has parabolic Hausdorff dimension $\leq n-1$.*

Remark 14.2. Notice that a static plane would have parabolic hausdorff dimension $n+2$.

As suggested by the previous sections, the key will be ruling out various tangent cones of high symemetry and/or multiplicity. Recall from remark key-idea of section stratification: if we can rule out (quasi-)static planes of multiplicity > 1, and static cones of the form $R^{n-1} \times$ (1-d minimal cone), or $R^{n-2} \times$ (2-d minimal cone), then general stratification theorem will prove the desired partial regularity.

A useful fact which can simplify many of the arguments in Brian's papers is

Theorem 14.3. *For a smooth, compact MCF $M(t)$ with $H > 0$, we have*

$$\min_{M(t)} \frac{\lambda_1}{H}$$

is increasing in time, and is strictly increasing if $\lambda_1 < 0$.

Proof. Follow directly by Hamilton's max principle for tensors. Or by observing that

$$\partial_t \lambda_1 - \Delta \lambda_1 \geq |A|^2 \lambda_1$$

in the viscosity sense. □

Consequently, for each translator M_λ, either $\frac{\lambda_1}{H} \geq 0$ or $\min \frac{\lambda_1}{H}$ is attained at the boundary. I claim that as $\lambda \to \infty$,

$$\min_{M_\lambda} \frac{\lambda_1}{H} \to \min_{M(0)\times R} \frac{\lambda_1}{H} > -\infty.$$

This follows because convergence is smooth near $M(0) \times R$, by the easy Brakke regularity theorem and classical short-time existence/uniqueness. So we can assume $\frac{\lambda_1}{H} \geq \lambda > -\infty$ for our smooth approximators.

14.4 one-sided-minimization

The most important property of mean convex flows is the following "one-sided-minimization" property.

Theorem 14.5 (one-sided minimizing)**.** *Let $t \in [0,t) \mapsto K(t)$ be the moving regions associated to a mean convex Brakke flow. Suppose $K(t) \subset \tilde{K} \subset K(0)$. Then $|\partial K(t)| \leq |\partial \tilde{K}|$. PICTURE.*

Proof. Let $\tilde{K}$ minimize $|\partial \tilde{K}|$ in the class

$$\{\Omega : K(t) \subset \Omega \subset K(0)\}.$$

Clearly it suffices to prove $|\partial K(t)| \leq |\partial \tilde{K}|$.

By the strong barrier principle $K(t) \subset \mathrm{int} K(0)$, so by mean convexity of $K(0)$ we must have $\tilde{K} \subset \mathrm{int} K(0)$ also. Away from $K(t)$ $\partial \tilde{K}$ is a minimal surface.

Now flow $K(0)$ until $K(\tau)$ just touches $\tilde{K}$. If $\tau = t$ then $\tilde{K} = K(t)$ and we are done. Otherwise $K(\tau)$ touches $\tilde{K}$ away from $K(t)$, and by using the approximators M_λ if necessary, we can assume $\partial K(\tau)$ is smooth. Therefore by the strong maximum principle (Solomon-White?) we must have $\tilde{K} = K(\tau)$. Now use that $|\partial K(t)|$ is decreasing in t. □

In particular, the one-sided minimization implies uniform volume bounds at all scales.

Corollary 14.6. *For any ball $B_r(x) \subset K(0)$, we have*

$$|K(t) \cap B_r(x)| \leq |\partial B_r|.$$

Here is a more refined version of Corollary REF.

Theorem 14.7. *Let $B(x,r) \subset K(0)$, and suppose*

$$\partial K(t) \cap B(x,r) \subset \mathrm{slab}(\epsilon r).$$

We have the following possibilities:

- *if $(K(t) \cap B) \sim \mathrm{slab}$ has 1 component, then $|\partial K(t) \cap B| \leq (1 + C\epsilon)\omega_n r^n$.*
- *if $K(t) \cap B \subset \mathrm{slab}$ (i.e. $(K(t) \cap B) \sim \mathrm{slab}$ has 0 components), then $|\partial K(t) \cap B| \leq (2 + C\epsilon)\omega_n r^n$.*
- *if $(K(t) \cap B) \sim \mathrm{slab}$ has 2 components, then $|\partial K(t) \cap B| \leq C\epsilon\omega_n r^n$.*

PICTURES

Proof. In each case, let $\tilde{K}$ be the region which agrees with $K(t)$ outside $B(x,r)$, and has boundary indicated by bold in the associated picture. The one-sided minimization property implies that

$$|\partial K(t) \cap B| \leq |\partial \tilde{K} \cap B| \leq (k + C\epsilon)\omega_n r^n,$$

where $k = 0, 1, 2$ depending on the indicated scenario. □

We therefore only have to worry about planes of multiplicity ≤ 2.

Corollary 14.8. *If M is a (quasi-)static plane obtained as a limit flow, then M has multiplicity either 0, 1 or 2. Further, if the multiplicity is 2, then the mean curvature points "in", in the sense that the interior region of the limit sequence is bounded by smaller and smaller slabs.*

Towards our stratification result, one-sided-minization immediately rules out (q-)static cones with codimension 1 or 2 spines.

Proposition 14.9. *A (q-)static, non-planar union of half-planes cannot be obtained as a blow-up limit flow.*

Proof. Suppose a static $R^{n-1} \times C$ is obtained as blow-up limit, where $C = \partial K$ is a 1-dimensional cone. Since $R^{n-1} \times C$ is obtained via a blow-up sequence, it is globally one-sided-minimizing. I claim that C is globally one-sided minimizing. Given $\tilde{K} \supset K$, with $\tilde{K} \sim K \subset B_R$, we have that

$$\left(B_\Lambda \times \tilde{K}\right) \cup \left((R^{n-1} \sim B_\Lambda) \times K\right) \supset R^{n-1} \times K.$$

And therefore $\Lambda^{n-1}|\partial K \cap B_R| \leq \Lambda^{n-1}|\partial\tilde{K} \cap B_R| + c_n\Lambda^{n-2}\text{diam}(\tilde{K})$. Taking $\Lambda \to \infty$ gives $|\partial K \cap B_R| \leq |\partial\tilde{K} \cap B_R|$.

Since C is not a line we can find two rays bounding a wedge W, with wedge angle $< \pi$ and $W \subset K^c$. But this contradicts one-sided-minimization of C. (actually to be entirely rigorous one would probably want to apply this logic sufficiently far along the blow-up sequence). □

Corollary 14.10. *A (q-)static, non-planar $R^{n-2} \times (2d\text{-cone})$ cannot be obtained as a blow-up limit flow.*

Proof. Otherwise, as in theorem REF, we would have that (cone) $\cap\, \partial B_1$ would be a geodesic network in the sphere. At any junction we can blow-up to obtain a non-planar union of half-planes, contradicting proposition REF. So (cone)$\cap \partial B_1$ is a great circle with multiplicity 1 or 2, and we are planar. □

Remark 14.11. We've ruled out tangent flows of the form $R^{n-1} \times$ (1-d cone) and $R^{n-2} \times$ (2-d cone). We've also shown that planes can have multiplicity at most 2. Therefore to prove the sing$\mathcal{M} \subset \mathcal{S}_{n-1}(\mathcal{M})$ we only need to rule out (q-)static planes of multiplicity 2 as tangent flows.

In general we have the following useful theorem.

Theorem 14.12. *If M is a one-sided-minimizing minimal variety, then* $\text{reg}(M)$ *is stable.*

Proof. Just look at variations in the good direction. □

14.13 separation theorem

The key fact we need to apply a stratification result is to show that singular points can disappear in a limit. Or conversely, we need to show that if a sequence $\mathcal{M}_i$ approaches a regular $\mathcal{M}$, of *any* multiplicity, then the convergence is smooth. Of course since smooth convergence is local, we only need to consider when $\mathcal{M}$ is a (q-)static plane. Further, since we know mult-1 convergence is smooth by Brakke's theorem, it suffices to show that the $\mathcal{M}_i$ break into a sub-collection $\mathcal{M}_i'$, each converging with mult-1.

In the mean-convex case we know that $\mathcal{M}$ can have multiplicity at most 2, so our life is further simplified. We will therefore work towards proving the crucial "separation" theorem: if $\mathcal{M}_i$ weakly limit to a mult-2 (q-)static plane, then for large i the $\mathcal{M}_i$ separate into disjoint sheets of mult 1.

Recall that if $V_i \to V$ as Brakke flows, then by monotonicity the spacetime supports converge locally in the hausdorff sense.

Define for any set X the "half-Reifenberg" distance

$$\text{th}(X, 0, r) = \inf_{V^m} \sup_{x \in X \cap B_r(0)} \text{dist}(x, V \cap B_r(0)).$$

The infimum is over affine m-planes V. Also, write $B_r^{sp}(X = (x,t)) = B_r(x) \times (t - r^2, t + r^2)$. Here is the picture of the following theorem: if we are in a sufficiently small slab, and our distance to zero is controlled in the parabolic scale, then our distance must be controlled in an absolute scale. One should view this as an effective version of the "vanishing hole" theorem, which says that a weak set flow supported in a surface minus a hole, must vanish instantly.

Q: Can we replace this with $\inf_V d_H(X \cap B_r^{sp}, (V \times R) \cap B_r^{sp})$? There seem to be technicalities.

Theorem 14.14. *$\exists \epsilon = \epsilon(\gamma, \Lambda, \beta, n)$ so that if $\mathcal{M}$ is a nested, eternal MCF, with $\Theta(\mathcal{M}) \leq \Lambda$, satisfying*

$$\mathrm{th}(K(-r^2), 0, r) \leq \epsilon \text{ for } r \geq 1, \quad \limsup_{r \to \infty} \frac{\mathrm{dist}(K(r^2), 0)}{r} \leq \beta,$$

then necessarily $\cap_t K(t)$ must meet B_γ.

Remark 14.15. By the nested property we have immediately $\mathrm{th}(K(t), r) \leq \epsilon$ for any $t \geq -r^2$.

Proof. Suppose not. Then we have a sequence $\mathcal{M}_i$ of nested flows satisfying the above hypotheses with $\epsilon_i \to 0$, and have the property that

$$\lim_{t \to \infty} \mathrm{dist}(K(t), 0) \geq \gamma.$$

Choose a sequence of times t_i so that:

$$R_i = \mathrm{dist}(K_i(t_i), 0) \geq \gamma/2,$$

and

$$2\beta \geq (1 + 1/i) \frac{\mathrm{dist}(K_i(t_i), 0)}{\sqrt{t_i}} \geq \sup_{t \geq t_i} \frac{\mathrm{dist}(K_i(t), 0)}{\sqrt{t}}.$$

Since both of these requirements improve as t increases, we can certainly satisfy both.

Define the rescaled flows

$$\tilde{\mathcal{M}}_i = \mathcal{D}_{1/R_i}(\mathcal{M}_i - (0, t_i)).$$

For any $\rho \geq 2/\gamma$, we have $\mathrm{th}(K_i(t_i), \rho R_i) \leq \epsilon_i$, and hence

$$\mathrm{th}(\tilde{K}_i(0), \rho) \leq \epsilon_i.$$

Our second requirement on t_i ensures that, for $t \geq 0$, we have

$$\begin{aligned} 1 &\leq \mathrm{dist}(\tilde{K}_i(t), 0) \\ &\leq (1 + 1/i) \frac{R_i}{\sqrt{t_i}} \sqrt{t + R_i^{-2} t_i} \\ &\leq (1 + 1/i) \sqrt{4\beta^2 t + 1}. \end{aligned}$$

Since we have uniformly bounded density, we can pass to a subsequential limit $\tilde{\mathcal{M}}_i \to \tilde{\mathcal{M}}$, obtaining a nested MCF flow in the limit. Write $\tilde{\mathcal{K}}$ for the corresponding interior. By the previous considerations, we have

$$\mathrm{th}(\tilde{K}(0), \rho) = 0$$

for every $\rho \geq 2/\gamma$, and therefore $\tilde{K}(0)$ is contained in a plane. But conversely, we have for $t \in (0, 1)$,

$$1 \leq \mathrm{dist}(\tilde{K}(0), 0) \leq \sqrt{4\beta^2 + 1}.$$

So although $\tilde{\mathcal{M}} \cap \{t > 0\}$ is supported in a plane, and avoid the unit ball, it is non-vanishing. This is a contradiction. □

Theorem 14.16. *$\exists \epsilon = \epsilon(n)$ so that if $\mathcal{M}$ be an eternal limit of smooth mean-convex flows, with $\Theta(\mathcal{M}) \leq 2$, and the property that for every $r \geq 1$ we have*

$$\mathrm{th}(K(-r^2), r) \leq \epsilon, \quad \frac{\mathrm{dist}(K(r^2), 0)}{r} \leq 1,$$

then $\mathcal{M}$ is a union of (possibly coincident) multiplicity 1 planes, with $\mathcal{K}$ being the region in between.

Proof. By the previous theorem we know (provided $\epsilon \leq \epsilon_0$) that $\cap_t K(t)$ is non-empty. Therefore $\Sigma = \partial(\cap_t K(t))$ is: a non-empty, one-sided minimizing minimal surface, whose tangent cone at infinity lies in an ϵ-slab. We elaborate: we know $\partial \cap_t K(t)$ is minimal as it coincides with any time slice of the static flow $\lim_t(\mathcal{M} - (0, t))$. The one-sided minimizing property follows since it can be realized as the limit of mean-convex flows. The

By Schoen-Simon, taking ϵ sufficiently small, Σ must be a union of planes. We elaborate. By theorem REF we know that $\mathcal{H}^{n-2}(\mathrm{sing}\Sigma) = 0$. For any $r \geq 1$ we have $\mathrm{th}(\Sigma, 0, r) \leq \epsilon$. So (provided $\epsilon \leq \epsilon_0$) Schoen-Simon says that away from the singular set Σ decomposes into disjoint graphs $u_1, \ldots, u_k$, with an estimate like

$$\sup_{B_{r/2}(0)} \left(|Du_i| + r|D^2 u_i|\right) \leq c\epsilon.$$

Taking $r \to \infty$ gives that the u_i are linear. So $k = 2$ since $\Theta_{eucl}(\Sigma) \leq 2$, and by disjointness we have (after rotation) $Du_1 = Du_2 = 0$. So in fact Σ is regular, and is the union of two (possibly coincident) planes.

Therefore, there is a plane P so that $P \subset K(t)$ for all t. Write $M(t) = M_1(t) \cup M_2(t)$, where M_i are the components on either side of P. We know that each $\Theta(\mathcal{M}_i) \geq 1$, by virtue of being non-empty, but but at the same time $\Theta(\mathcal{M}) = \Theta(\mathcal{M}_1 \cup \mathcal{M}_2) \leq 2$. So in fact $\Theta(\mathcal{M}_i, x, \cdot) \equiv 1$ for every $x \in \mathcal{M}_i$. Therefore each $\mathcal{M}_i$ is a plane parallel to P. □

The above theorem allows us to prove that, if we converge weakly to a multiplicity 2 plane, then at very small scales we split into two disjoint graphs, each very close to a plane. The basic idea is to choose our rescalings intelligently so we always remain a fixed distance away from a mult-2 plane.

Theorem 14.17. *Suppose $\mathcal{M}_i$ are smooth, mean-convex, with the property that*

$$d_H(\mathcal{K}_i \cap B_1^{sp}, (P \times R) \cap B_1^{sp}) \to 0,$$

for some plane P.

Then there is a blow-up sequence $\mathcal{D}_{1/\rho_i}\mathcal{M}_i$ converging smoothly to a pair of parallel planes, so that $\mathcal{D}_{1/\rho_i}\mathcal{K}_i$ converges to the region in between.

Proof. Let ϵ be any number small than the ϵ from the above theorem. Choose ρ_i to be the least number ≤ 1 so that for $r \in [\rho_i, 1]$ we have

$$\mathrm{th}(K_i(-r^2), r) \leq \epsilon, \quad \frac{\mathrm{dist}(K_i(r^2), 0)}{r} \leq 1.$$

Since the $\mathcal{M}_i$ are smooth we clearly have $\rho_i > 0$, and by hypothesis we know that $\rho_i \to 0$.

Now set $\tilde{\mathcal{M}}_i = \mathcal{D}_{1/\rho_i}\mathcal{M}_i$. Take a subsequential limit $\tilde{\mathcal{M}}_i \to \tilde{\mathcal{M}}$. This $\tilde{\mathcal{M}}$ will be a limit of mean-convex flows, which satisfies for every $r \geq 1$

$$\mathrm{th}(\tilde{K}(-r^2), r) \leq \epsilon, \quad \frac{\mathrm{dist}(\tilde{K}(r^2), 0)}{r} \leq 1 \tag{2}$$

Further, one of these inequalities is an equality at $r = 1$.

By the one-sided minimization, we have $\limsup_i \Theta(\mathcal{M}_i, 0, r) \leq 2 + \delta(r)$, for $\delta(r) \to 0$ as $r \to 0$. Therefore, for any R we have

$$\begin{aligned} \Theta(\tilde{\mathcal{M}}, 0, R) &= \lim_i \Theta(\mathcal{M}_i, 0, \rho_i R) \\ &\leq \limsup_i \Theta(\mathcal{M}_i, 0, r) \quad \text{for any } r \in (0, 1) \\ &\leq 2 + \delta(r). \end{aligned}$$

Taking $r \to 0$ gives the estimate $\Theta(\tilde{\mathcal{M}}) \leq 2$.

Therefore by theorem REF $\tilde{\mathcal{M}}$ is either a plane of multiplicity 1 or 2, or is a union of mult-1 planes. But equality in either of EQREF implies $\tilde{\mathcal{M}}$ cannot be supported in a plane. So $\tilde{\mathcal{M}}$ is a union of two planes, and therefore by Brakkes theorem $\tilde{\mathcal{M}}_i$ converge smoothly to $\tilde{\mathcal{M}}$. □

Let $S_i(t)$ be the set of centers of open balls in $K(t)$, whose closure touch $\partial K(t) = M(t)$ at 2 or more points. Let $S_i = \{(x, t) : x \in S_i(t)\}$ be the spacetime collection of these centers.

Theorem 14.18. *Suppose $\mathcal{M}_i$ are smooth, mean-convex, with the property that*

$$d_H(\mathcal{K}_i \cap B_4^{sp}, (P \times R) \cap B_4^{sp}) \to 0,$$

for some plane P.

Then for i sufficiently large, $S_i \cap B_1^{sp}$ is a smooth hypersurface in spacetime, which divides $\mathcal{M}_i$ into two, non-empty components $\mathcal{M}_i' \cup \mathcal{M}_i''$.

In particular, any convergent subsequence of $\mathcal{M}_i$ on B_1^{sp} converges smoothly to a multiplicity 2 plane.

Proof. Suppose, on the contrary, that for each i we can find a point $x_i \in B_2^{sp}$, radius r_i, so that

$$S_i \cap B_r^{sp}(x_i) \text{ is not a hypersurface, for every } r \in (0, r_i].$$

Set $\tilde{\mathcal{M}}_i = \mathcal{M}_i - x_i$. Then $\tilde{\mathcal{M}}_i$ satisfies the assumptions of theorem REF, and so for sufficiently large i we have some radius r_i for which $\mathcal{K}_i \cap B_{r_i}^{sp}(x_i)$ splits into the region between two smooth, disjoint graphs, with small $C^{1,1}$ norm and C^0 norm tending to 0 with i. Therefore $S_i \cap B_{r_i}(x_i)^{sp}$ is a smooth hypersurface. This is a contradiction.

Let $\mathcal{M}_i$ be a convergent subsequence in B_2^{sp}. We have $\mathcal{M}'_i \to \mathcal{M}'$ and $\mathcal{M}''_i \to \mathcal{M}''$ as Brakke flows. By assumption,

$$d_H(\mathcal{M}' \cap B_1^{sp}, (P \cap R) \cap B_1^{sp}) = d_H(\mathcal{M}'' \cap B_1^{sp}, (P \cap R) \cap B_1^{sp}) = 0,$$

and so $\mathcal{M}'$, $\mathcal{M}''$ are both static planes of some multiplicity. However at each $t \in (-1, 1)$ we have by one-sided minimization that

$$1 + 1 \leq \Theta^n_{eucl}(M'(t), 0) + \Theta^n_{eucl}(M''(t), 0) \leq \Theta_{eucl}(M'(t) \cup M''(t), 0, 1) = 2.$$

And so each is a multiplicity 1 plane. By Brakke's regularity we deduce that $\mathcal{M}'_i$ and $\mathcal{M}''_i$ converge smoothly. □

Theorem 14.19. *Suppose $\mathcal{M}_i$ are smooth, mean-convex, and limit to a (q-)static, multiplicity-2 plane P. Then $\mathcal{M}_i$ converge smoothly as disjoint graphs.*

Proof. Since the supports converge in the Hausdorff sense, we for any ball $B_r(X)^{sp}$ (away from the time of vanishing, if P is quasi-static)

$$d_H(\mathcal{K}_i \cap B_r^{sp}(X), (P \times R) \cap B_r^{sp}(X)) \to 0.$$

So by the separation theorem REF, $\mathcal{M}_i$ converge smoothly on $B_r^{sp}(X)$. Therefore $\mathcal{M}_i$ converge smoothly on compact subsets of spacetime (if P is static) or compact subsets of $R^{n+1} \times (-\infty, 0)$ (if P is quasi-static). □

Corollary 14.20. *Let $\mathcal{D}$ be the collection of mean-convex $\mathcal{M}$ with the property that any point with a planar (q-)static tangent flow is smooth. Then $\mathcal{D}$ is closed under convergence of Brakke flows.*

Proof. Let $\mathcal{D}$ be Let $\mathcal{M}_i$ be the weak limit of $\mathcal{M}_{i,j}$, as $j \to \infty$. Choose a diagonal sequence j_i, so that $\mathcal{M}_{i,j_i}$ converge to a mult-2 plane. Then for i sufficiently large, the convergence is smooth. But this holds for any $j'_i \geq j_i$, and so for each fixed i, the convergence $\mathcal{M}_{i,j} \to \mathcal{M}_i$ is also smooth. □

Remark 14.21. Recall that due to non-fattening, any mean-convex $\mathcal{M}$ can be obtained through Ilmanen's elliptic regularization. Since every elliptic approximator lies in $\mathcal{D}$, this shows every mean-convex $\mathcal{M}$ lies in $\mathcal{D}$ also.

Corollary 14.22. *No multiplicity 2 (quasi-)static plane can occur as a tangent flow of a smooth mean-convex flow $\mathcal{M}$.*

Proof. Otherwise, convergence would be smooth, and hence at a small scale $\mathcal{M}$ would separate into (disjoint, by strict nesting) graphs. But then the argument of theorem curve-shortening-version would give a contradiction. □

For a general mean-convex $\mathcal{M}$ one can obtain a mult-2 tangent flow, since $\mathcal{M}$ need not be strictly nested. However in this case we get a good global picture of how $\mathcal{M}$ must look. See the next section.

14.23 classification of blow-ups

We've bounded the size of singular points. Let us now consider their structure.

Theorem 14.24. *Let $\mathcal{M}$ be a mean-convex flow, and let $\tilde{\mathcal{M}}$ be an eternal blow-up limit flow. We have the following:*

A) If $\tilde{\mathcal{M}}$ is a (q-)static minimal cone M, then $\tilde{\mathcal{M}}$ a mult-1 or 2 plane.

B) If $\tilde{\mathcal{M}}$ is a non-static self-shrinker, then it must be $S^k \times R^{m-k}$ (with mult 1).

C) If $\tilde{\mathcal{M}}$ is a (q-)static minimal variety M, then either $\tilde{\mathcal{M}}$ is a mult-1 or 2 plane, or it is a pair of mult-1 parallel planes.

Remark 14.25. Notice we do not require $\tilde{\mathcal{M}}$ to arise as a tangent flow.

Proof. Recall that we can (and will) assume $\tilde{\mathcal{M}}$ is actually the limit of smooth mean-convex flows.

Part A. At almost every point we have a tangent plane of multiplicity 1 or 2. By theorem REF convergence to these points is smooth. Since λ_1/H is bounded below in our sequence, these points must be flat. So M is a polyhedral cone. If C is itself not flat can (iteratively) blow-up and obtain union of half-planes. But this contradicts proposition REF.

Part B. Blow-up at any point in $M(-1)$ to get a static cone. This must be a multiplicity 1 or 2 plane by part A, and hence convergence is globally smooth. So $\tilde{\mathcal{M}}$ is a smooth, mean-convex self-shrinker, and is therefore $S^k \times R^{m-k}$. If $k > 0$ it is multiplicity 1, otherwise it violates one-sided-minimizing (K would become the infintesimal region between the two-sheets).

Part C. As before blowing up at any point gives a mult 1 or 2 plane, and hence convergence is globally smooth. By the bound on λ_1/H we get that M must be flat and embedded. Since the tangent cone at infinity is a plane of mult ≤ 2, we get that M is either: a plane of mult 1 or 2, or a union of two mult-1 planes. □

Since we've ruled out (q-)static multiplicity 2 planar tangent flows, we obtain

Corollary 14.26. *Any tangent flow to $\mathcal{M}$ is a mult-1 generalized cylinder $S^k \times R^{n-k} = \partial B^k \times R^{n-k}$.*

We work towards ruling out (q-)static mult-2 planar *limit* flows. Here is the general strategy: Rescale (contract) limiting sequence so we're fixed distance away from mult-2 plane in appropriate sense, but tangent flow at origin (of new limit) is still mult-2 plane. Then in neighborhood $B_r \times (-\infty, 0)$ new limit agrees with static, mult-2 min surface. Take $t = -\infty$ translational limit, get static guy M of mult ≥ 2. but since we're "far" away from mult-2 plane, M isn't mult-2 plane either. so $\Theta(M) > 2$, but contradicts classification that M must be pair of planes.

We need the following effective versions of the separation theorem and the no-mult-2-tangent-flows theorem.

Theorem 14.27 (effective separation)**.** *There is an $\epsilon_2(n)$ so that if $\mathcal{M}$ is a limit flow, and*

$$d_H(\mathcal{K} \cap B_4^{sp}, (P \times R) \cap B_4^{sp}) \leq \epsilon_2,$$

then $\mathcal{M} \cap B_1^{sp}$ splits into smooth graphs $f \leq g$ over P, of small C^2-norm, and $\mathcal{K} \cap B_1^{sp}$ is the region in between them.

Remark 14.28. Since f, g satisfy the graphical MCF equation, $f < g$ unless $f \equiv g$.

Proof. Otherwise, there is a (WLOG smooth) counterexample sequence $\mathcal{M}_i$, satisfying the hypothesis of the separation theorem REF. But this contradicts the conclusion of theorem REF. □

Corollary 14.29. *Let $\mathcal{M}$ be a blow-up limit flow, and suppose at 0 some tangent flow of $\mathcal{M}$ is a (q-)static, mult-2 plane. Then in (parabolic) neighborhood of 0 $\mathcal{M}$ agrees with a static minimal hypersurface $S \ni 0$ with multiplicity 2.*

Of course if $\mathcal{M}$ were actually smooth, a mult-2 (q-)static tangent flow would be impossible.

Proof. In a sufficiently small (parabolic) neighborhood of 0 $\mathcal{M}$ satisfies the conditions of theorem effective-separation. So $\mathcal{M}$ locally splits into two graphs $f \leq g$.

If the tangent flow is a static, mult-2 plane, $\mathcal{M}$ is locally graphical in a spacetime neighborhood, and $f = g$ at 0. So $f \equiv g$. If the tangent flow is (q-)static, then the argument of theorem curve-shortening-version would give a contradiction unless $f \equiv g$ also.

Since $\mathcal{M}$ is nested $f \equiv g$ must be indepedent of time. Since the blow-up is planar through the origin, S must contain 0. □

Corollary 14.30. *Let $\mathcal{M}$ be a blow-up limit flow, so that some tangent flow at 0 is a (q-)static mult-2 plane. Then there is a minimal hypersurface $S \ni 0$, and a neighborhood r, so that*

$$\Theta(\mathcal{M}, \cdot) \geq 2 \text{ on } (S \cap B_r) \times (-\infty, 0).$$

Remark 14.31. Notice this means that in the negative time limit we can't disappear.

Proof. By theorem REF, in some parbolic spacetime neighborhood $B_r \times (-r^2, 0)$ $\mathcal{M}$ must agree with a static minimal surface $S \times (-r^2, 0)$ of multiplicity 2.

By theorem no-half-planes and no-2d-cones, the spacetime set $\mathcal{Z}$ of points in $\mathcal{M}$ which do not admit a (q-)static planar tangent flow has $\dim_{par} \mathcal{Z} \leq n-1$. Therefore if $\pi\mathcal{Z}$ is the space projection, then $\dim_{par} \pi\mathcal{Z} \leq n-1$ also. So $\mathcal{H}^n(S \cap \pi\mathcal{Z}) = 0$.

We show $\Theta(\mathcal{M}, \cdot) = 2$ on $(S \sim \pi\mathcal{Z}) \times (-\infty, 0)$. By USC this will suffice to prove the theorem.

Pick a point $y \in S \sim \pi\mathcal{Z}$, and let

$$t^* = \inf\{t : \Theta(\mathcal{M}, (y,t)) = 2\}.$$

Suppose, towards a contradiction, that $t^* > -\infty$. Let $\mathcal{M}'$ be any tangent flow at (y, t^*). $\mathcal{M}'$ cannot be quasi-static or static mult-1 since $\Theta(\mathcal{M}', \cdot) \geq 2$ on positive times. Neither can $\mathcal{M}'$ be q-static of mult-2, as otherwise in a parbolic neighborhood of (y, t^*) $\mathcal{M}$ agrees with a mult-2 static hypersurface, contradicting our choice of t^*. Therefore $y \in S \cap \pi\mathcal{Z}$, a contradiction. $\square$

We wish to have an effective statement about when the tangent flow is multiplicity-2. Let us define a more global notion of "close to multiplicity-2 (q-)static plane." Given a flow $\mathcal{M}$, define

$$\phi(\mathcal{K}) = \inf_s \{\inf_V d_H(\mathcal{K} \cap (B_{1/s} \times (-\infty, 0)), (V \cap B_{1/s}) \times (-\infty, 0)) \leq s\}.$$

As $\phi \to 0$, $\mathcal{M}$ looks more and more like on bigger and bigger scales like a (q-)static mult-2 plane.

Corollary 14.32. *There is an $\epsilon_2(n)$ so that: if $\mathcal{M}'$ is a tangent flow to (limit) flow $\mathcal{M}$, and*

$$\phi(\mathcal{M}') < \epsilon_2,$$

then $\mathcal{M}'$ is a (q-)static mult-2 plane.

Proof. Suppose the statement is false. Then there is a sequence of tangent flows $\mathcal{M}_i'$, which are not mult-2 planes, but with $\phi(\mathcal{M}_i') \to 0$. By the effective separation Theorem REF, and after a spatial rotation as necessary, we know that on increasingly large subsets of $R^n \times (-\infty, 0)$ the $\mathcal{M}_i'$ splits into graphs $g_i \leq f_i$.

Each f_i, g_i solve the MCF equation, and f_i decreases with time, while g_i increases in time. If $f_i = g_i$ then by parabolic-dilation-invariance $\mathcal{M}_i'$ must be a mult-2 plane. So we must have $g_i < f_i$ for all i.

Define

$$u_i = \frac{f_i - g_i}{f_i(0,-1) - g_i(0,-1)}.$$

The u_i is a positive solution to a linear PDE, with coefficients tending in C^∞ to the heat equation as $i \to \infty$. By the Harnack inequality ("earlier times bounded by later times") and the fact that u_i is decreasing, means that u_i is uniformly bounded above and below on compact sets of $R^n \times (-\infty, 0)$.

So we can take a smooth limit $u_i \to u$, where $u : R^n \times (-\infty, 0) \to R$ is a positive, decreasing solution to the heat equation, with

$$u(0, -1) = 1.$$

We can apply the Harnack inequality up to $t = 0$. Therefore if we consider $u - \inf u$, we must have $u \equiv \inf u$ being constant, non-zero.

But since each $\mathcal{M}_i$ is self-similar, we must have

$$u(\lambda x, \lambda^2 t) = \lambda u(x, t).$$

This is a contradiction. □

We are finally ready to show that mult-2 (q-)static planes cannot occur as a blow-up limit.

Theorem 14.33. *Let $\mathcal{M}$ be a smooth, mean-convex flow, and take $\mathcal{M}_i$ a blow-up sequence. Then $\mathcal{M}_i$ cannot limit to a mult-2 (q-)static plane.*

Proof. Suppose $\mathcal{M}_i$ converges to a (q-)static mult-2 plane $\mathcal{M}'$. Then necessarily $\phi(\mathcal{K}_i) \to 0$.

Since $\mathcal{M}$ is smooth, there is a biggest $\lambda_i \leq 1$ so that $\phi(\mathcal{D}_{\lambda_i}\mathcal{K}_i) \geq \eta/2$. Notice that in particular we have,

$$\phi(\mathcal{D}_s \mathcal{D}_{\lambda_i} \mathcal{K}_i) \leq \eta/2 \quad \forall s \in [1, 1/\lambda_i].$$

By assumption $\lambda_i \to 0$.

Take a subsequential limit $\mathcal{D}_{\lambda_i}\mathcal{K}_i \to \mathcal{K}'$. This is a Brakke flow satisfying $\phi(\mathcal{K}') \geq \eta/2$, but $\phi(\mathcal{D}_s\mathcal{K}') \leq \eta/2$ for any $s \geq 1$. Therefore by Theorem REF any tangent flow at the origin is a mult-2 (q-)static plane. So by theorem tangent-flow-to-min we have that $\Theta(\mathcal{M}', \cdot) \geq 2$ on $S \times (-\infty, 0] \in (0, 0)$, where S is some minimal surface.

Notice that since $\Theta(\mathcal{M}', 0) = 2$, but $\mathcal{M}'$ is not a multiplicity 2 plane (otherwise $\phi(\mathcal{K}')$ would be 0), we have $\Theta(\mathcal{M}') > 2$. Take any tangent flow at $t = -\infty$, to obtain a static minimal surface M, with density $\Theta_{eucl}(M) > 2$. This contradicts our classification theorem REF. □

This proves a kind of local non-collapsing. Given $x \in \mathcal{M}$, define

$$\mu(x, t) = \frac{1}{\text{inscribed radius in } \mathcal{K}(t)}.$$

Corollary 14.34. *Let $\mathcal{M}$ be a mean-convex MCF, so that $\mathcal{M}(0)$ is smooth and T is the first singular time. Then*

$$\frac{\mu}{H} \leq C \quad \text{on } [0, T)$$

Proof. Suppose, towards a contradiction, we had a sequence of points X_i for which $\mu/H \to \infty$. We can assume $X_i \to X$. Necessarily, $X \notin \text{reg}\mathcal{M}$, as otherwise we would have $\mu(X) = \infty$, which contradicts the maximum principle.

So we must have $X = (x, T)$, and $H(X_i) \to \infty$. Let $\mathcal{M}_i = \mathcal{D}_{H(X_i)}(\mathcal{M} - X_i)$. This is a blow-up sequence with $\mu(\mathcal{M}_i, 0) \to \infty$. Pass to a limit $\mathcal{M}_i \to \mathcal{M}'$, and by Hausdorff convergence we must have $\Theta(\mathcal{M}', 0) \geq 2$. But $\Theta(\mathcal{M}') = 2$, and so $\mathcal{M}'$ is a mult-2 (q-)static plane. This is a contradiction. $\square$