Math 60670 Midterm

Due by 11:59pm Friday March 22. Send your exam to `nedelen@nd.edu`. You may use Lee's and do Carmo's books, the class notes, and previous homeworks and midterms from this class, but no other resources. You are allowed to quote theorems/lemmas/corollaries from the book or class or previous homework *that we or the book have proven.* You are not allowed to quote statements from the books that are given without proof (e.g. exercises).

Q1: In class we showed there always exists a unique torsion-free, metric-compatible linear connection on (M, g) (the Levi-Civita connection). Consider the sets $\mathcal{C}_1$ consisting of the connections which are torsion-free but not necessarily metric-compatible, and $\mathcal{C}_2$ consisting of the connections which are metric-compatible but not necessarily torsion-free. For each $i = 1, 2$, show that $\mathcal{C}_i$ is in bijection with a certain set of $(1, 2)$-tensor fields on M, and find a 0-th order characterization for these tensors (i.e. without using any derivatives or connections).

Q2: Let X be a smooth vector field in (M, g), and let $\phi_t(p)$, $t \in I$, be the flow of X. Given a $(0, 2)$-tensor field T on M, define the Lie derivative of T w.r.t. X to be the $(0, 2)$-tensor defined by

$$(\mathcal{L}_X T|_p)(Y, Z) := \lim_{t \to 0} \frac{1}{t}(T|_{\phi_t(p)}(D\phi_t|_p Y, D\phi_t|_p Z) - T|_p(Y, Z)).$$

A. Prove that $\mathcal{L}_X$ obeys the following product rule: if $X, Y, Z \in \mathcal{X}(M)$, then

$$X(T(Y, Z)) = (\mathcal{L}_X T)(Y, Z) + T(\mathcal{L}_X Y, Z) + T(Y, \mathcal{L}_X Z),$$

where $\mathcal{L}_X Y = [X, Y]$.

B. X is called a Killing field if the diffeomorphism ϕ_t is an isometry for all $t \in I$. Show that

$$\begin{aligned} X \text{ is a Killing field } &\iff \mathcal{L}_X g = 0 \\ &\iff < \nabla_Y X, Z > + < Y, \nabla_Z X >= 0 \text{ for all vectors } Y, Z. \end{aligned}$$

(here ∇ is of course the Levi-Civita connection w.r.t. g)

C. If $X = \text{grad} f$ for some function f, show that X is Killing $\iff$ $\nabla^2 f = 0$.

D. Prove that along a geodesic γ, a Killing field X satisfies $< X, \gamma' >= const.$ (Aside: if M is a rotationally symmetric surface, then taking X to be the Killing field generating rotations gives a more geometric proof of Clairaut's relation, which you derived in HW5.)

Q3: Let (M^n, g) be a complete, non-compact Riemannian manifold. We say M has m ends if there are compacts sets $K_i \subset K_{i+1}$ such that $M = \cup_i K_i$ and $M \setminus K_i$ has precisely m connected components for all $i >> 1$. A geodesic line $\gamma : \mathbb{R} \to M$ is a geodesic parameterized by arclength such that $d(\gamma(s), \gamma(t)) = |t - s|$ for all $s, t \in \mathbb{R}$ (i.e. $\gamma|_I$ is minimizing for every finite interval I). Show that if M has more than one end, then M contains a geodesic line. Does the converse hold?