

Q1 let $\bar{\nabla} = \text{LC connection}$

define $A : \{\text{connections}\} \rightarrow \{(2,1)\text{-tensors}\}$

$$\nabla \mapsto A_{\nabla}(X, Y) = \nabla_X Y - \bar{\nabla}_X Y$$

check A well-defined

A_{∇} trivially \mathbb{R} -linear, and C^∞ linear in X

check C^∞ -linear in Y : $A_{\nabla}(X, fY)$

$$= \nabla_X(fY) - \bar{\nabla}_X(fY)$$

$$= X(f)Y - X(f)Y + fA_{\nabla}(X, Y) \quad \checkmark$$

A_{∇} bijective (since $\bar{\nabla} + A = \text{connection}$ for any $(2,1)$ -tensor A)

choose $E_i = \text{ON frame @ } p$, write $A_{\nabla}(E_i, E_j) = A_{ij}^k E_k$

$$\nabla \text{ torsion free} \Leftrightarrow \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

$$= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

o.p

$$\Leftrightarrow \nabla_X Y - \bar{\nabla}_X Y - (\nabla_Y X - \bar{\nabla}_Y X) = 0 \quad \forall X, Y$$

$$\Leftrightarrow A_{\nabla}(X, Y) = A_{\nabla}(Y, X) \quad \forall X, Y$$

$$\Leftrightarrow A_{ij}^k = A_{ji}^k \quad \forall i, j, k$$

$$\nabla \text{ metric compatible} \Leftrightarrow X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

$$= X(g(Y, Z)) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z)$$

o.p

$$\Leftrightarrow g(\nabla_X Y - \bar{\nabla}_X Y, Z) + g(Y, \nabla_X Z - \bar{\nabla}_X Z) = 0 \quad \forall X, Y, Z$$

$$\Leftrightarrow g(A_0(x, y, z) + g(y, A_0(x, z))) = 0$$

$$\Leftrightarrow A_{ij}^k = -A_{ik}^j \quad \forall i, j, k$$

Q5.1. choose coords near p

$$\Rightarrow D_{e_i} Y|_p = Y^i|_p \frac{\partial e_i^k(p)}{\partial x^k} \partial_k$$

$$\begin{aligned} \circ (Z \times T)(Y; Z) &= \frac{\partial}{\partial t} \Big|_{t=0} \left(T_{ij} \Big|_{e_i} Y^p \frac{\partial e_i^j}{\partial x^r} z^q \Big|_p \frac{\partial e_j^i}{\partial x^s} \right) \\ &= (\partial_t T_{ij}) Y^i z^j + T_{ij} Y^p \partial_r X^k z^i \\ &\quad + T_{ij} Y^i z^q \partial_q X^k \\ &= \partial_t (T_{ij} \Big|_{e_i} Y^i \Big|_{e_i} z^i \Big|_{e_i}) \\ &\quad - T_{ij} X^k \partial_k Y^i z^j - T_{ij} Y^i X^k \partial_k z^j \\ &\quad + T_{ij} Y^k \partial_k X^i z^j + T_{ij} Y^i z^k \partial_k X^j \\ &= \partial_t (T(Y, Z)) - T([X, Y], Z) \\ &\quad - T(Y, [X, Z]) \end{aligned}$$

Q. $X = \text{Killing} \Leftrightarrow \mathcal{L}_X g = 0 \quad \forall t, p$

$$\Leftrightarrow \frac{d}{dt} \mathcal{L}_t g = 0 \quad \forall t, p$$

$$\Leftrightarrow \frac{d}{dt} g_{e_i e_j} (D_{e_i} Y, D_{e_j} Z) = 0 \quad \forall t, p, Y, Z$$

small

$$e_{t+s} = e_t \circ \phi_s \Leftrightarrow \frac{d}{ds} \Big|_{s=0} g_{e_i e_j} (D_{e_i \circ \phi_s} Y, D_{e_j \circ \phi_s} Z) = 0 \quad \forall t, p, Y, Z$$

$$\Leftrightarrow \left(\sum_{i,j} g_{ij} \right) (D_{x_i} Y, D_{x_j} Z) = 0 \quad \forall Y, Z \in \mathcal{V}_p \quad \forall p$$

$$\Leftrightarrow \sum_{i,j} g_{ij} = 0 \quad \forall p \quad \text{since } D_{x_i} \text{ non-singular} \\ \text{at } p \text{ defies -}$$

$$\Leftrightarrow X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) = 0 \\ \forall X, Y, Z \in \mathcal{X}(M)$$

$$\Leftrightarrow g(D_X Y, Z) + g(Y, D_X Z) \\ - g(D_X Y - D_Y X, Z) - g(Y, D_X Z - D_Z X) = 0$$

$$\Leftrightarrow g(D_Y X, Z) + g(Y, D_Z X) = 0 \quad \forall X, Y, Z \in \mathcal{X}(M)$$

C. if $X = \text{grad } f$

the X Killing $\Leftrightarrow g(D_Y \text{grad } f, Z) + g(Y, D_Z \text{grad } f) = 0$

$$\Leftrightarrow \Delta_{Y,Z}^2 f + \Delta_{Z,Y}^2 f = 0$$

$$\Leftrightarrow 2 \Delta_{Y,Z}^2 f = 0$$

D. δ geodesic $\Rightarrow \delta' \langle \delta', X \rangle = \langle \cancel{D_{\delta'} \delta'}, X \rangle + \langle \delta', D_{\delta'} X \rangle \\ = \frac{1}{2} (\langle \delta', D_{\delta'} X \rangle + \langle D_{\delta'} X, \delta' \rangle) \\ = 0$

Q: WLOG space $M(k_i)$ has m components $\forall i$

choose $p_i, q_i \in M(k_i)$ lying in different ads

then $\text{dist}(p_i, k_i) \rightarrow \infty$ and $\text{dist}(q_i, k_i) \rightarrow \infty$

[side note: there was mistake in setup of problem, one needs to assume $\text{dist}(p, M(k_i)) \rightarrow \infty$ for any $p \in M(k_i)$

and if $\gamma_i =$ minimizing geodesic $p_i \rightarrow q_i$

necessarily $\gamma_i \cap k_i \neq \emptyset \forall i$

\Rightarrow WLOG $\gamma_i: [a_i, b_i] \rightarrow M$ PB&L and $\gamma_i(0) \in k_i$

now (after passing to subseq) $\gamma_i(0) \rightarrow p \in k_i$

and $\gamma_i'(0) \rightarrow v \in T_p M$ (since $|\gamma_i'(0)|=1$)

\Rightarrow by continuous dependence on initial conditions $\gamma_i \rightarrow \gamma_{p,v} =: \gamma$ in C_{loc}^1

$\forall T, \gamma_i|_{[-T, T]} \rightarrow \gamma_{p,v}|_{[-T, T]} \in C^1$

$\Rightarrow L\gamma_i|_{[-T, T]} \rightarrow L\gamma$

"
 $d(\gamma_i(-T), \gamma_i(T))$ (for $i \gg 1$)

\downarrow
 $d(\gamma(-T), \gamma(T))$ (since $\gamma_i(-T) \rightarrow \gamma(-T)$, etc.)

$\Rightarrow \gamma|_{[-T, T]}$ mmz $\forall T \Rightarrow \gamma =$ geodesic line