Q1: let x^i = normal coords for M @ p

$v, w \in T_pM$ unit vectors $\longrightarrow w = w^i \partial_i|_p$

$\hookrightarrow t \mapsto t(v + sw)$ geodesic $\forall s$

$\Rightarrow \frac{\partial}{\partial s}\big|_{s=0} = V(t) = tw^i \partial_i\big|_{tv}$ = Jacobi field $\longrightarrow$ and

$V(0) = 0$

$V'(0) = w$

let $f(t) = |V(t)|^2 = t^2 g_{ij}(tv) w^i w^j$

$\hookrightarrow f' = 2\langle V, V' \rangle$

$f'' = 2|V'|^2 - 2R(V, \gamma', \gamma', V)$

$f''' = 4\langle V', V'' \rangle - 2(\nabla_t R)(V, \gamma', \gamma', V) - 4R(V', \gamma', \gamma', V)$

$= -8R(V', \gamma', \gamma', V) - 2(\nabla_t R)(V, \gamma', \gamma', V)$

$f'''' = -2(\nabla_t^2 R)(V, \gamma', \gamma', V) - 4(\nabla_t R)(V', \gamma', \gamma', V)$

$-8(\nabla_t R)(V', \gamma', \gamma', V) - 8R(V', \gamma', \gamma', V')$

$+ 8R(V, \gamma', \gamma', R(V, \gamma')\gamma')$

$f^{(5)} = -20(\nabla_t R)(V', \gamma', \gamma', V') +$ (terms linear in V)

So) $f(0) = 0$, $f'(0) = 0$, $f''(0) = 2|w|^2$,

$f'''(0) = 0$, $f^{(4)}(0) = -8R(w, v, v, w)$

$f^{(5)}(0) = -20(\nabla_v R)(w, v, v, w)$

So we can Taylor expand:

$$t^2 g_{ij}(tv) w^i w^j = f(t) = t^2 |w|^2 - \frac{8t^4}{4\cdot 3\cdot 2} R(w,v,v,w)$$

$$- \frac{20 t^5}{5\cdot 4\cdot 3\cdot 2} (\nabla_v R)(v,w,w,v)$$

$$+ O(t^6)$$

$$\Rightarrow g_{ij}(tv) w^i w^j = |w|^2 - \frac{t^2}{3} R(w,v,v,w) - \frac{t^3}{6} (\nabla_v R)(v,w,w,v) + O(t^4)$$

since w arbitrary, $|v|=1$

$$\Rightarrow g_{ij}(x) = \delta_{ij} - \frac{R_{ikej}}{3} x^k x^l - \frac{(\nabla_n R)_{ikej}}{6} x^n x^k x^l + O(|x|^4)$$

B. recall that $\det(e^A) = e^{\operatorname{tr} A}$ $\forall$ matrices A

$\hookrightarrow$ if $|t|$ is small, then $I + tB = e^{A(t)}$

for $A(t) = \ln(I + tB)$

$$= tB - \frac{t^2 B^2}{2} + \frac{t^3 B^3}{3} + O(t^4)$$

$$\Rightarrow \det(I + tB) = e^{\operatorname{tr} A(t)}$$

$$= 1 + t \operatorname{tr} B + \frac{t^2}{2}\left(-\operatorname{tr}(B^2) + (\operatorname{tr} B)^2\right) + O(t^3 B^3)$$

now $g_{ij}(tv) = \delta_{ij} + t B_{ij}$, $B_{ij} = -\frac{t}{3} R_{ikej} v^k v^l$

(v unit vector) $\quad - \frac{t^2}{6} (\nabla_m R)_{ikej} v^m v^k v^l$

$+ O(t^3)$

$$\Rightarrow \det(g_{ij}(tv)) = 1 - \frac{t^2}{3} Ric(v,v) - \frac{t^3}{6} (\nabla_v Ric)(v,v) + O(t^4)$$

$$\Rightarrow dV|_{tv} = \sqrt{\det g_{ij}(tv)} \qquad \text{since } \sqrt{1-x} = 1 - \frac{x}{2} + O(x^2)$$

$$= 1 - \frac{t^2}{6} Ric(v,v) - \frac{t^3}{12} (\nabla_v Ric)(v,v) + O(t^4)$$

we then get:

$$Vol_g(B_r(p)) = \int_{B_r(0)} \sqrt{\det g_{ij}(x)}\, dx$$

$$= \int_0^r r^{n-1} dr \int_{S^{n-1}} d\theta \sqrt{\det g_{ij}(r\theta)}$$

$$= \int_0^r r^{n-1} dr \int_{S^{n-1}} d\theta \left(1 - \frac{r^2}{6} Ric(\theta,\theta) - \frac{r^3}{12} (\nabla_\theta Ric)(\theta,\theta) + O(r^4)\right)$$

$$= \omega_n r^n - \frac{r^{n+2}}{(n+2)6} \int_{S^{n-1}} Ric(\theta,\theta)\, d\theta - \frac{r^{n+3}}{(n+3)\cdot 12} \int_{S^{n-1}} (\nabla_\theta Ric)(\theta,\theta)\, d\theta$$

$$+ O(r^{n+4})$$

from HWS, $\displaystyle\int_{S^{n-1}} Rc(\theta,\theta)\,d\theta = \frac{|S^{n-1}|}{n}\, Scal$

$= \omega_n\, Scal$

If $A(\theta) = -\theta$ = antipodal map

$\Rightarrow \displaystyle\int_{S^{n-1}} (\nabla_\theta Rc)(\theta,\theta)\,d\theta = \int_{A(S^{n-1})} (\nabla_\theta Rc)(\theta,\theta)\,d\theta$

$= \displaystyle\int_{S^{n-1}} (\nabla_{-\theta} Rc)(-\theta,-\theta)\,d\theta = -\int_{S^{n-1}} (\nabla_\theta Rc)(\theta,\theta)\,d\theta$

$\Rightarrow \displaystyle\int_{S^{n-1}} (\nabla_\theta Rc)(\theta,\theta)\,d\theta = 0$

So $\displaystyle Vol_g(B_r(p)) = \omega_n r^n - \frac{\omega_n r^{n+2}}{6(n+2)}\, Scal(p) + O(r^{n+4})$

Q2. A: Bianchi identity $\Rightarrow \nabla_s R_{ijkl} + \nabla_i R_{jskl} + \nabla_j R_{sikl}$

trace over j and k

$$\Rightarrow \nabla_s R_{ij}{}^{j}{}_{l} + \underbrace{\nabla_i R_{js}{}^{j}{}_{l}}_{\substack{\shortparallel \\ -\nabla_i R_{sj}{}^{j}{}_{l}}} + \nabla_j R_{si}{}^{j}{}_{l} = 0$$

$$\Rightarrow \nabla_s Ric_{il} - \nabla_i Ric_{sl} + \nabla_j R_{si}{}^{j}{}_{l} = 0$$

trace over i and l

$$\Rightarrow \nabla_s Scal - \nabla_i Ric_s{}^{i} - \nabla_i Ric_s{}^{i} = 0$$

$$\Rightarrow \nabla_i Ric_p{}^{i} = \frac{1}{2}\nabla_p (Scal)$$

Einstein condition $\Rightarrow$ LHS $= \nabla_i (g^{ia} Ric_{pa})$

$= \nabla_i (\lambda g^{ia} g_{pa})$

$= \nabla_i (\lambda \delta_{ip})$

$= \nabla_p \lambda$

$= \frac{1}{n} \nabla_p Scal$ since $Scal = tr\, Ric$

$= tr(\lambda g)$

$= n\lambda$

$$\Rightarrow \frac{1}{2}\nabla_p Scal = \frac{1}{n}\nabla_p Scal$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{n}\right)\nabla_p Scal = 0 \Rightarrow \partial_p Scal = 0 \quad \text{if } n \geq 3$$

$\Rightarrow Scal = n\lambda =$ locally constant

B. spse $n=3 \Rightarrow Ric = \frac{1}{3} S g$ for $S =$ Scal

$\Rightarrow R_{1221} + R_{1331} = \frac{1}{3} S$ + (for e_1, e_2, e_3 ON basis)

and $R_{2112} + R_{2332} = \frac{1}{3} S$ +

$R_{3113} + R_{3223} = \frac{1}{3} S$ −

$\Rightarrow 2R_{1221} = \frac{1}{3} S$

$\Rightarrow \mathrm{Sect}(e_1, e_2) = \frac{1}{6} S$

but e_1, e_2, e_3 arbitrary $\Rightarrow \mathrm{Sect} \equiv \frac{1}{6} S =$ const.

Q3. $\{z = x^2+y^2\} = M$ = closed subset of $\mathbb{R}^3$

$\Rightarrow M$ = complete

choose coord chart $F(x,y) \mapsto (x, y, x^2+y^2)$

$\Rightarrow \partial_x F = (1, 0, 2x)$

$\partial_y F = (0, 1, 2y)$

$$\Rightarrow \text{induced metric } g = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ & \boxed{\begin{matrix} 1+4x^2 & 4xy \\ 4xy & 1+4y^2 \end{matrix}} \end{array}$$

$\partial^2_{xx} F = (0, 0, 2)$ and unit normal $\nu = \dfrac{(-2x, -2y, 1)}{\sqrt{1+4x^2+4y^2}}$

$\partial^2_{xy} F = (0, 0, 0)$

$\partial^2_{yy} F = (0, 0, 2)$

$\Rightarrow$ second fundamental form $h = \dfrac{1}{\sqrt{1+4x^2+4y^2}} \boxed{\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}}$

Gauss equation $\Rightarrow$ $\text{Sect} = \dfrac{\det h}{\det g} = \dfrac{\frac{4}{1+4x^2+4y^2}}{(1+4x^2)(1+4y^2) - 16x^2y^2}$

$= \dfrac{4}{(1+4x^2+4y^2)^2} > 0$