Math 60670 Homework 8

Due Thursday April 11.

Problem 1: A Riemannian submanifold $M \subset (\bar{M}, \bar{g})$ (with the induced metric $g = \bar{g}|_{TM}$), is called totally geodesic if for every $p \in M$, $v \in T_pM$, the $\bar{g}$-geodesic γ with initial conditions $\gamma(0) = p$, $\gamma'(0) = v$ lies in M. Show the following are equivalent:

A. M is totally geodesic,

B. Every g-geodesic in M is also a $\bar{g}$-geodesic in $\bar{M}$,

C. The second fundamental form of M vanishes.

Problem 2: Let $M \subset \mathbb{R}^3$ be the catenoid, which is the surface of revolution obtained by revolving the curve $x = \cosh z$ around the z-axis. Show that M has zero mean curvature, i.e. show the trace of the second fundamental form is zero.

Problem 3: A geodeisc triangle in a Riemannian 2-manifold (M^2, g) is a domain Ω with piecewise-smooth boundary $\partial\Omega$ consisting of three geodesics meeting at three vertices. If M has constant Gauss curvature K, show that the sum of interior angles of any geodesic triangle is $\pi + KA$, where A is the area of Ω.

Problem 4: Let $M \subset (\bar{M}, \bar{g})$ be a compact, embedded Riemannian submanifold, and let NM be the normal bundle, i.e. the subbundle of $T\bar{M}$ consisting of vectors $v \in T_x\bar{M}$ for $x \in M$, $v \perp T_xM$. Let $(NM)_\epsilon = \{V \in NM : |V| < \epsilon\}$, and let $M_\epsilon = \{x \in \bar{M} : dist(x, M) < \epsilon\}$.

A. Show that for all small $\epsilon > 0$, the exponential map restricted to $(NM)_\epsilon$ is a diffeomorphism from $(NM)_\epsilon$ to M_ϵ. For these ϵ, the open set M_ϵ is called a tubular neighborhood of M.

B. If $r(x)$ denotes the distance from $x \in \bar{M}$ to M, show that r^2 is a smooth function on any tubular neighborhood M_ϵ. Give an example where r^2 is not smooth on all of $\bar{M}$.